Statistical 3D Shape Analysis Using Square-Root Normal Fields

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A shape is the form of an object or its external boundary, outline, or external surface, as opposed to other properties such as color, texture, or material composition.
A shape is the form of an object or its external boundary, outline, or external surface, as opposed to other properties such as color, texture, or material composition.

Mathematician and statistician David George Kendall writes:[2]

In this paper 'shape' is used in the vulgar sense, and means what one would normally expect it to mean. [...] We here define 'shape' informally as 'all the geometrical information that remains when location, scale[3] and rotational effects are filtered out from an object.'
What is shape

• Shape is not affected by some shape-preserving transformations

Shape is what is left when differences, which can be attributed to translations, scale, and rotations have been filtered out

David G. Kendall (1984) – first to introduce statistics into shape analysis
Why shape is important?

Biomedical

[Boyer et al. 11]

3D modeling

[Chaudhuri et al. 11]

Archaeology

[Huang et al. 06]
Problem statement

- Given a population of 3D objects, we want to
  - Study and model the shape variability within the population (mean shape, modes of variation)
Problem statement

• Given a population of 3D objects, we want to
  – Study and model the shape variability within the population (mean shape, modes of variation)
  – Build statistical models that describe the population

Automatically generated random human shapes in random poses
Problem statement

- Given a population of 3D objects, we want to
  - Study and model the shape variability within the population (mean shape, modes of variation)
  - Build statistical models that describe the population
  - Analyze & model deformations and growth patterns
    - How the 3D shape of the brain evolves with Alzheimer?
    - Typical growth curve of a foetus?
    - Differences in the growth of human body across countries
    - Correlation or causality relations (between the spatial distribution of Kebab shops with human body shape)
Let’s focus on the mean shape
Let’s focus on the mean shape
Let’s focus on the mean shape

• The registration problem
Let’s focus on the mean shape

- The registration problem
Let’s focus on the mean shape

- The registration problem
Let’s focus on the mean shape

• The registration problem
Things are not always so simple ...
Things are not always so simple ...

Wrong alignment
Correct correspondence

Correct alignment
Wrong correspondence

Average shape

Average shape
Let’s focus on the mean shape

• First problem - We need correct registration
  – Correct alignment and correct correspondences under rigid transformations & non-rigid deformations

  Rigid transformations (do not change shape)
  Non-rigid deformations (do change the shape)
Let’s focus on the mean shape

- The space of shapes is not Euclidean

(a) Linear path

$$(1 - t)f_1 + tf_2$$
Let’s focus on the mean shape

- The space of shapes is not Euclidean

(a) Linear path

\[(1 - t)f_1 + tf_2\]

(b) Natural deformation
Let’s focus on the mean shape

- The space of shapes is not Euclidean

(a) Linear path

$(1 - t)f_1 + tf_2$
Let’s focus on the mean shape

- Second problem: we need an appropriate non-linear metric

(a) Linear path
$$(1 - t)f_1 + tf_2$$

(b) Natural deformation
In this presentation

• Background and motivation
• Problem 1: Elastic registration
  – Surface representation
  – Re-parameterization and registration
• Problem 2: what is the right metric for comparing shapes
  – The elastic metric for 3D shape analysis
  – The Square Root Normal Field (SRNF) representation
• Problem 3: the SRNF inversion problem
• Applications
Representation of surfaces

• Parameterized surfaces

\[ f : \mathbb{S}^2 \rightarrow \mathbb{R}^3 \]

\[ s = (u, v) \rightarrow f(s) = (x(s), y(s), z(s)) \]

Spherical parameterization of closed Genus-0 surfaces
Remove shape-preserving transformations

• Parameterized surfaces

\[ f : S^2 \rightarrow \mathbb{R}^3 \]

\[ s = (u, v) \rightarrow f(s) = (x(s), y(s), z(s)) \]

• Normalize all shapes for translation and scale
  – Translate all the shapes so that their centre of mass is at origin
  – Scale all the shapes to have unit surface area

• How about rotation and correspondences?
Parameterization provides registration

Initial parameterization (not optimal)

Re-parameterization of \( f_2 \) (optimal)

Diffeomorphism
Parameterization provides registration

• Find the optimal rotation and diffeomorphism to apply to $f_2$ so that it becomes as close as possible to $f_1$

\[ d_S([f_1], [f_2]) = \min_{O \in SO(3), \gamma \in \Gamma} d(f_1, O(f_2 \circ \gamma)). \]

Correspondence
Let’s focus on the mean shape

• First problem - We need correct registration

\[ d_S([f_1],[f_2]) = \min_{O \in SO(3), \gamma \in \Gamma} d(f_1, O(f_2 \circ \gamma)). \]

- Assume correspondences are given
  • Find optimal rotation (SVD decomposition if \( d \) is Euclidean)

- Assume optimal rotation is given
  • Solve for optimal correspondence, i.e. re-parameterization or diffeomorphism \( \gamma \)
  • Involves search over the space of all possible diffeomorphisms

- Repeat many times the two steps above
Let’s focus on the mean shape

• First problem - We need correct registration

\[ d_S([f_1],[f_2]) = \min_{O \in SO(3), \gamma \in \Gamma} d(f_1, O(f_2 \circ \gamma)). \]

Correspondence

• Second problem
  – What is \( d \)? How do we measure distances between surfaces?
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The space of shapes is not Euclidean

(a) Linear path

\[(1 - t)f_1 + tf_2\]

(b) Natural deformation
Elastic shape metric for comparing surfaces

The difference between two surfaces is the amount of bending and stretching needed to align one surface onto the other.
Elastic shape metric for comparing surfaces

The difference between two surfaces is the amount of bending and stretching needed to align one surface onto the other

Three ways of quantifying bending

- Differences in the orientation of normal vectors
- Differences in the surface curvatures
- Differences in the Second Fundamental Forms (II)
Elastic shape metric for comparing surfaces

The difference between two surfaces is the amount of bending and stretching needed to align one surface onto the other.

Three ways of quantifying bending

- Differences in the orientation of normal vectors
- Differences in the surface curvatures
- Differences in the Second Fundamental Forms (II)

Two ways of quantifying stretch (elasticity)

- Differences in local surface area
- Differences in the First Fundamental Form (the metric)
The general elastic metric

The difference between two surfaces is the amount of bending and stretching needed to align one surface onto the other.

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Two ways of quantifying stretch (elasticity)

- Differences in local surface area
- Differences in the First Fundamental Form (the metric)
Simplified elastic metric

The difference between two surfaces is the amount of bending and stretching needed to align one surface onto the other.

Three ways of quantifying bending

- Differences in the orientation of normal vectors
- Differences in the surface curvatures
- Differences in the Second Fundamental Forms (II)

Two ways of quantifying stretch (elasticity)

- Differences in local surface area
- Differences in the First Fundamental Form (the metric)
The partial elastic metric (Jermyn et al.)

- **Surface bending**
  - Change in the orientation of the normal vectors
  - Normal to a surface $f$ at a point $s$

\[
 n(s) = \frac{\partial f}{\partial u}(s) \times \frac{\partial f}{\partial v}(s) \quad \tilde{n}(s) = \frac{n(s)}{|n(s)|}
\]

- **Surface stretching**
  - Change in local area of $f$ at $s$: \[ r(s) = |\tilde{n}(s)| \]

- A surface $f$ can then be represented with $(r, \tilde{n})$
Partial elastic shape metric

The difference between $f$ and $g$ is the amount of bending and stretching needed to align one surface onto the other.

\[
d((r_1, \tilde{n}_1), (r_2, \tilde{n}_2)) = \alpha \int \frac{\partial r_1(s) \partial r_2(s)}{r(s)} ds + \beta \int \langle \partial \tilde{n}_1(s), \partial \tilde{n}_2(s) \rangle r(s) ds.
\]
Elastic shape metric for comparing surfaces

The difference between \( f \) and \( g \) is the amount of bending and stretching needed to align one surface onto the other.

\[
d((r_1, \tilde{n}_1), (r_2, \tilde{n}_2)) = \alpha \int \frac{\partial r_1(s) \partial r_2(s)}{r(s)} ds + \beta \int (\partial \tilde{n}_1(s), \partial \tilde{n}_2(s)) r(s) ds.
\]

Penalizes stretching
Elastic shape metric for comparing surfaces

The difference between $f$ and $g$ is the amount of bending and stretching needed to align one surface onto the other.

$$d((r_1, \tilde{n}_1), (r_2, \tilde{n}_2)) = \alpha \int \frac{\partial r_1(s) \partial r_2(s)}{r(s)} ds + \beta \int (\partial \tilde{n}_1(s), \partial \tilde{n}_2(s)) r(s) ds.$$
The difference between $f$ and $g$ is the amount of bending and stretching needed to align one surface onto the other.

- Not Euclidean
- Very complex to evaluate and use for shape statistics
- Computationally very expensive

\[ d((r_1, \tilde{n}_1), (r_2, \tilde{n}_2)) = \alpha \int \frac{\partial r_1(s) \partial r_2(s)}{r(s)} ds + \beta \int (\partial \tilde{n}_1(s), \partial \tilde{n}_2(s)) r(s) ds. \]
Square-Root Normal Field (SRNF)

- SRNF representation of surfaces introduced by Jermyn et al. ECCV2012

\[ q(s) = \sqrt{r(s)} \hat{n}(s) = \frac{n(s)}{\sqrt{r(s)}} = \frac{n(s)}{|n(s)|^{1/2}}. \]

- For \( \alpha = \frac{1}{4} \), and \( \beta = 1 \), the elastic metric reduces to the Euclidean distance between SRNF representations of surfaces

\[
\begin{align*}
    d(f_1, f_2) &= \frac{1}{4} \int \frac{\partial r_1(s) \partial r_2(s)}{r(s)} ds + \int \langle \partial \hat{n}_1(s), \partial \hat{n}_2(s) \rangle r(s) ds \\
    &= \|q_1 - q_2\|^2 \\
    &= \int \|q_1(s) - q_2(s)\|^2 ds.
\end{align*}
\]
SRNF linearizes the manifold of shapes

Map all the shapes to the SRNF space

Space of parameterized surfaces (non-linear)
- Geodesic paths and distances are hard to compute
- Difficult to perform statistics

SRNF space is Euclidean
- Straight lines correspond to optimal deformations (geodesics)
- Standard linear statistics
SRNF linearizes the manifold of shapes

Perform all the analysis in the space of SRNFs

Space of parameterized surfaces
(non-linear)

SRNF space is Euclidean

- Mean = \( (q_1 + q_1) / 2 \)
- Linear interpolation: \( tq_1 + (1-t)q_2 \)
- Statistics: standard PCA
SRNF linearizes the manifold of shapes

Map the results back to the space of surfaces

Space of parameterized surfaces (non-linear)

SRNF space is Euclidean
- Mean = $(q_1 + q_1) / 2$
- Linear interpolation: $tq_1 + (1-t)q_2$
- Statistics: standard PCA
Applications of SRNFs

- Comparison of shapes
- Elastic registration of shapes
- Computing geodesics (optimal deformation of one surface onto another)
- Transferring deformations
- Statistical shape analysis
  - Mean shape and modes of variations
  - Characterizing populations with probability distributions
  - Generating arbitrary 3D shapes
Using the SRNF

- Comparing 3D shapes

\[
d(f_1, f_2) = \min_{O \in SO(3), \gamma \in \Gamma} d(f_1 - O(f_2 \circ \gamma))
\]

\[
= \min_{O \in SO(3), \gamma \in \Gamma} \|q_1 - O(q_2, \gamma)\|
\]

\[
O(q_2, \gamma) = (q_2 \circ \gamma) \sqrt{J_\gamma}
\]

where \(J_\gamma\) is the determinant of the Jacobian of \(\gamma\).
Using the SRNF

- Comparing 3D shapes

\[
d(f_1, f_2) = \min_{O \in SO(3), \gamma \in \Gamma} d(f_1 - O(f_2 \circ \gamma))
\]

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= \min_{O \in SO(3), \gamma \in \Gamma} \|q_1 - O(q_2, \gamma)\|
\]

\[
O(q_2, \gamma) = (q_2 \circ \gamma) \sqrt{J_{\gamma}}
\]

where \(J_{\gamma}\) is the determinant of the Jacobian of \(\gamma\).

- Elastic registration of 3D shapes

\[
(O^*, \gamma^*) = \arg\min_{O \in SO(3), \gamma \in \Gamma} d(f_1 - O(f_2 \circ \gamma))
\]

\[
= \arg\min_{O \in SO(3), \gamma \in \Gamma} \|q_1 - O(q_2, \gamma)\|
\]
Registration and classification results

• Registration results
  – Anatomical surfaces
  – Complex shapes

• Classification results
  – Generic 3D shapes (SHREC07 dataset)
  – Medical imaging
    • diagnosis of attention deficit hyperactivity disorder (ADHD)
Elastic registration of carpal bones
Registration results – anatomical 3D shapes

Elastic registration of carpal bones

\[ f_1 \]

\[ f_2 \]
Registration results – anatomical 3D shapes

Elastic registration of carpal bones
Elastic registration of carpal bones
Correspondence results – complex shapes

Isometric deformations

$L[F^*] = 0.1609$

$L[F^*] = 0.1369$

Correspondences are color-coded
Correspondence results – complex shapes

Elastic deformations
Correspondence results – complex shapes

Correspondence in the presence of missing parts

\[ L[F^*] = 0.0997 \]

\[ (L[F^*] = 0.1977) \]
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• Problem 3: the SRNF inversion problem
• Applications
Shape statistics using SRNFs

• The SRNF map should be invertible

Space of parameterized surfaces (non-linear)

SRNF space (linear)

• Mean = \( \frac{(q_1 + q_1)}{2} \)
• Linear interpolation: \( tq_1 + (1-t)q_2 \)
• Statistic: standard PCA
The SRNF map should be invertible

- We know how to compute SRNFs

\[ q(s) = \sqrt{r(s)} \tilde{n}(s) = \frac{n(s)}{\sqrt{r(s)}} = \frac{n(s)}{|n(s)|^{\frac{1}{2}}}. \]

- Their inverse is not unique and it does not have a closed analytical form (at least we don’t know it)

Good news

- We can invert it numerically
SRNF inversion

• Formulation
  – Given $q$, we want to find $f$ such that $\text{SRNF}(f) = Q(f)$ is as close as possible to $q$

$$E_0(f; q) = \min_{O, \gamma} \|Q(f) - O(q, \gamma)\|^2_2$$
SRNF inversion

• Formulation
  – Given $q$, we want to find $f$ such that $\text{SRNF}(f) = Q(f)$ is as close as possible to $q$

$$E_0(f; q) = \min_{O, \gamma} \|Q(f) - Q(q, \gamma)\|^2_2$$

  – Define the surface $f$ as the deformation of a reference surface $f_0$ (e.g. a sphere)

$$f = f_0 + w,$$

  – Parameterize the space of deformations with some orthonormal basis

$$w = \sum_{b \in \mathcal{B}} \alpha_b b,.$$
SRNF inversion

- **Formulation**
  
  - Given \( q \), we want to find \( f \) such that \( \text{SRNF}(f) = Q(f) \) is as close as possible to \( q \)

  \[
  E_0(f; q) = \min_{O, \gamma} \| Q(f) - O(q, \gamma) \|_2^2
  \]

  - Define the surface \( f \) as the deformation of a reference surface \( f_0 \) (e.g. a sphere)

  \[
  f = f_0 + w,
  \]

  - Parameterize the space of deformations with some orthonormal basis

  \[
  w = \sum_{b \in B} \alpha_b b,
  \]

  - General surfaces: spherical harmonic basis
  - Domain-specific data: use PCA basis
SRNF inversion

• Formulation

  – Given \( q \), we want to find \( f \) such that \( \text{SRNF}(f) = Q(f) \) is as close as possible to \( q \)

\[
E_0(f; q) = \min_{O, \gamma} \|Q(f) - O(q, \gamma)\|_2^2
\]

\[
E(w; q) = \min_{O, \gamma} \|Q(f_0 + w) - O(q, \gamma)\|_2^2,
\]
SRNF inversion by gradient descent
Multi-resolution SRNF representation

• Use spherical wavelet decomposition
Multi-resolution SRNF representation

- Use spherical wavelet decomposition
Multi-resolution SRNF representation

- Use spherical wavelet decomposition

\[ q^1 \quad q^2 \quad q^3 \quad q^4 \quad q^5 \]

Multiresolution SRNF
Multi-resolution SRNF representation

• Use spherical wavelet decomposition
Multi-resolution SRNF representation

- Use spherical wavelet decomposition
Multi-resolution SRNF representation

- Use spherical wavelet decomposition
Multi-resolution SRNF representation

- Use spherical wavelet decomposition
Some inversion results

(a) The target surfaces $f_o$.

\[ q_0 = Q(f_0) \]

\[ f^* = Q^{-1}(q_0) \]

(b) The reconstructed surfaces $f^*$.

(c) Pixel-wise errors $|f^*(s) - f_o(s)|$. 
Some inversion results

(a) The target surfaces $f_0$.

$$q_0 = Q(f_0)$$

$$Q^{-1}(q_0)$$

(b) The reconstructed surfaces $f^*$. 

(c) Pixel-wise errors $|f^*(s) - f_0(s)|$. 
Some inversion results

(a) The target surfaces $f_0$.

$\mathbf{q}_0 = Q(f_0)$

$\mathbf{f}^* = Q^{-1}(\mathbf{q}_0)$

(b) The reconstructed surfaces $\mathbf{f}^*$.

$\mathbf{f}^* = Q^{-1}(\mathbf{q}_0)$

(c) Pixel-wise errors $|\mathbf{f}^*(s) - f_0(s)|$.
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• Applications
Geodesic paths

• Map shapes to SRNF space

SRNF space (linear)
Geodesic paths

• Linear interpolation on SRNF space

\[ q_t = tq_1 + (1-t)q_2 \]
Geodesic paths

- Linear interpolation on SRNF space

SRNF space (linear)
Geodesic paths

• Map lines from SRNF space back to original space

Inverse SRNF

SRNF space (linear)
(a) Linear path $(1 - t)f_1 + tf_2$
(a) Linear path \((1 - t)f_1 + tf_2\)
Geodesic paths

(a) Linear path \((1 - t)f_1 + tf_2\)

(b) Geodesic path \(\alpha(t)\) by SRNF inversion
Geodesic paths

(a) Linear path \((1 - t)f_1 + tf_2\)
(registration computed with SRNF)
Geodesic paths

(a) Linear path \((1 - t)f_1 + tf_2\)
(registration computed with SRNF)

(c) Geodesic path using SRNF inversion proposed here.
Geodesic paths

(a) Linear path \((1 - t)f_1 + tf_2\)
(registration computed with SRNF)
Geodesic paths

(a) Linear path $(1 - t)f_1 + tf_2$
(registration computed with SRNF)

(d) Geodesic path using SRNF inversion proposed here.
Geodesic paths

(a) Linear path \((1 - t)f_1 + tf_2\) (registration computed with SRNF)

(b) Geodesic path using SRNF inversion proposed here.
Geodesic paths

(a) Linear path with SRNF registration.

(b) Geodesic path using SRNF inversion proposed here.
Geodesic paths

(a) Linear path with SRNF registration.

(b) Geodesic path using SRNF inversion proposed here
Symmetry analysis

Shape symmetrization and measure of asymmetry

Shape $f$
Symmetry analysis

Shape symmetrization and measure of asymmetry

Shape $f$

\[ \tilde{f} = H(\nu)f \]

(Reflection of $f$ with respect to an arbitrary plane)

\[ H(\nu) = (I - 2\frac{\nu\nu^T}{\nu^T\nu}) \]
Symmetry analysis

Shape symmetrization and measure of asymmetry

Shape $f$

$L(F^*) = 0.1535$

$\tilde{f} = H(\nu)f$

(Reflection of $f$ with respect to an arbitrary plane)

Length of the path is a measure of asymmetry
Symmetry analysis

- Shape symmetrization and measure of asymmetry

\[ L(F^*) = 0.0963 \]

\[ L(F^*) = 0.1189 \]
Deformation transfer

Source deformation
Deformation transfer

Source deformation

\(f_1\) \hspace{2cm} \(h_1\) \hspace{2cm} \(f_2\)
Deformation transfer

Source deformation

\[ f_1 \]  
\[ h_1 \]  
\[ f_2 \]  
\[ h_2 \]
Deformation transfer

• Parallel transport in the SRNF space
  – We are given $f_1, h_1, f_2$, we need to find $h_2$
  – Compute
    • $Q(f_1)$, $Q(h_1)$, $Q(f_2)$
    • $v = Q(h_1) - Q(f_1)$;
    • $q = Q(f_2) + v$
  – Invert $q$ to obtain $h_2$
Deformation transfer results

(a) Source deformation

\[ f_1 \quad h_1 \]
Deformation transfer results

(a) Source deformation

(b) Deformation transfer by linear extrapolation,

\[ h_2 = f_2 + \alpha(h_1 - f_1) \]
Deformation transfer results

(a) Source deformation

(b) Deformation transfer by linear extrapolation,
\[ h_2 = f_2 + \alpha(h_1 - f_1) \]

(c) Deformation transfer by SRNF inversion
\[ Q(h_2) = Q(f_2) + \alpha(Q(h_1) - Q(f_1)). \]
Deformation transfer results

(a) Source deformation $f_1$, $h_1$

(b) Deformation transfer by linear extrapolation,
$$h_2 = f_2 + \alpha(h_1 - f_1)$$

(c) Deformation transfer by SRNF inversion
$$Q(h_2) = Q(f_2) + \alpha(Q(h_1) - Q(f_1)).$$
Deformation transfer results

(a) Source deformation

(b) Deformation transfer by linear extrapolation,

\[ h_2 = f_2 + \alpha(h_1 - f_1) \]

(c) Deformation transfer by SRNF inversion

\[ Q(h_2) = Q(f_2) + \alpha(Q(h_1) - Q(f_1)). \]
Summary statistics

• Back to our mean shape
  – Given a set of surfaces $f_1, f_2, \ldots$
  – We want to compute the mean shape and the modes of variations

• Using SRNFs
  – Compute $Q(f_1), Q(f_2), \ldots$
  – Use Principal Component Analysis (PCA) in the SRNF space
  – Invert the mean and principal directions back to the surface space
Summary statistics results

(a) Shape.

Mean and modes of variation
Mean and modes of variation
Random human body shapes

(a) Shape.  
(b) Pose.

Mean and modes of variation  
Random human body shapes
Classification of shapes
Classification of shapes
Classification of shapes

Pose 12

Pose 3

Pose 7

Pose 5

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Classification of shapes – SHREC07

• MDS plots

Euclidean distance between surfaces before registration
Classification of shapes – SHREC07

- MDS plots

Euclidean distance between surfaces after registration

Euclidean distance in SRNF shape space
Summary

• **SRNF representation**
  – Efficient registration even under large elastic deformations
  – Linearizes the shape space
    • Perform standard analysis (using vector calculus) in the space of SRNFs
    • Map the results back to the space of surfaces

• **Modelling tasks become straight forward vector calculus operations**
  – Deformations, deformation transfer
  – Symmetrization
  – 3D shape generation
  – Statistical classification
  – Regressions
Limitations

• The elastic registration procedure requires parameterized surfaces
  – Closed genus-0 surfaces $\rightarrow$ spherical parameterization
  – Open surfaces (e.g. human faces) $\rightarrow$ disk parameterization
  – High genus surfaces are hard to parameterize

• Do not handle topological changes
  – E.g. when a bone erodes, a hole might appear.
  – It can be a parameterization issue
Limitations

• The elastic metric

  – SRNF is a special case for $\alpha = \frac{1}{4}$, $\beta = 1$

  $$d((r_1, \tilde{n}_1), (r_2, \tilde{n}_2)) = \alpha \int \frac{\partial r_1(s) \partial r_2(s)}{r(s)} ds + \beta \int \langle \partial \tilde{n}_1(s), \partial \tilde{n}_2(s) \rangle r(s) ds.$$ 

  – Ideally, we want to control the weight of each term
  – There is no such nice simplification for arbitrary $\alpha$ and $\beta$
Related publications


How about objects with structural variability?