Statistical Shape Analysis of Tree-like 3D Objects

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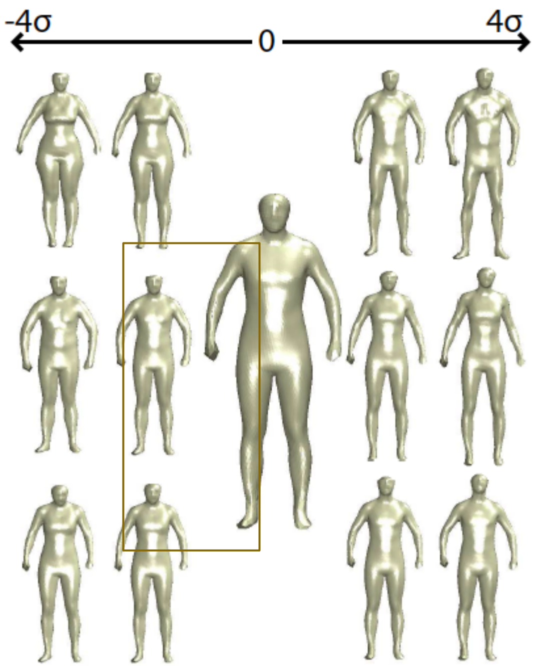
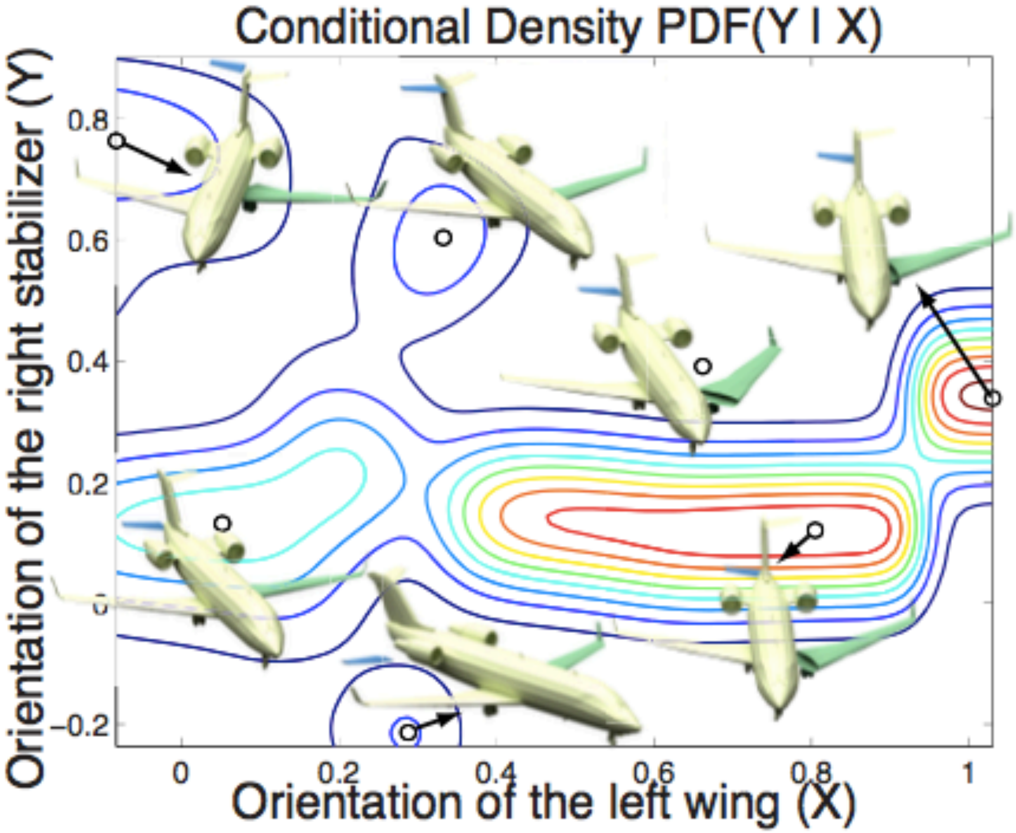
Statistical shape analysis



(a) Shape.

Statistical summaries
Mean and modes of variation



Represent variability
with statistical models

Analyze and model
growth and deformation

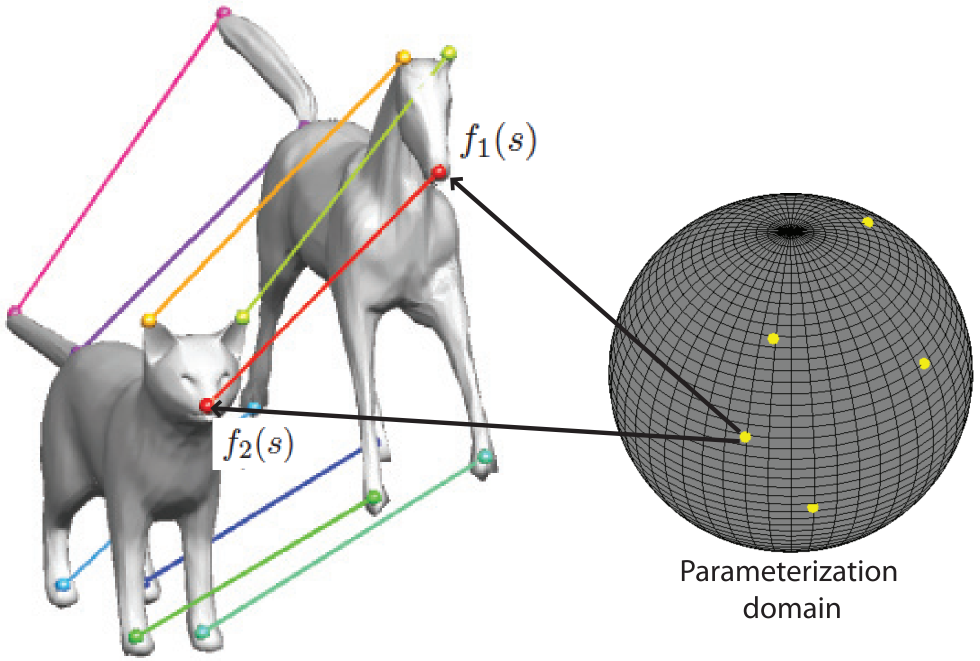
The building blocks

- A representation
- A metric for measuring dissimilarities (distances) between shapes
  - Is invariant to shape preserving transformations
  - Measures deformations that change shape
  - Is easy to compute
- A mechanism for computing correspondences and geodesics
  - A geodesic is an optimal sequence of deformations that align one shape onto another

Representation

$f : \mathbb{S}^2 \to \mathbb{R}^3$

s=(u, v) → $f(s) = (x(s), y(s), z(s))$



Correspondences = diffeomorphisms

Surface normals

$$n(s) = \frac{\partial f}{\partial u}(s) \times \frac{\partial f}{\partial v}(s)$$

$$\tilde{n}(s) = \frac{n(s)}{|n(s)|}$$

Area

$$r(s) = |n(s)|$$

Curvatures

…

Quantifying shape similarities and differences

- The difference between two surfaces is the amount of bending and stretching needed to align one surface onto the other

Different ways of quantifying bending

Differences in the orientation of normal vectors

Differences in the surface curvatures

Differences in the Second Fundamental Forms (II)

Different ways of quantifying stretch (elasticity)

Differences in local surface area

Differences in the First Fundamental Form (the metric)

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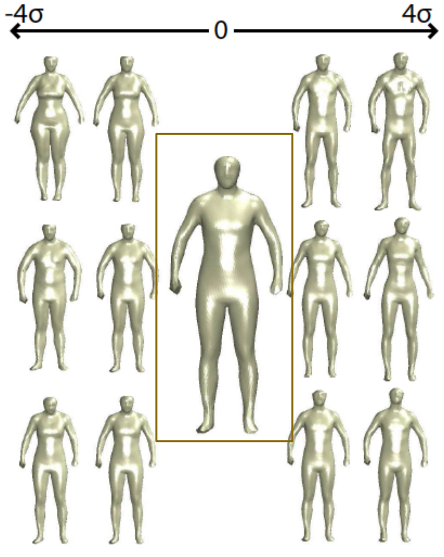
Differences in the First Fundamental Form (the metric)

Jermyn et al. (ECCV2012) Elastic Shape Matching of Parameterized Surfaces Using Square Root Normal Fields

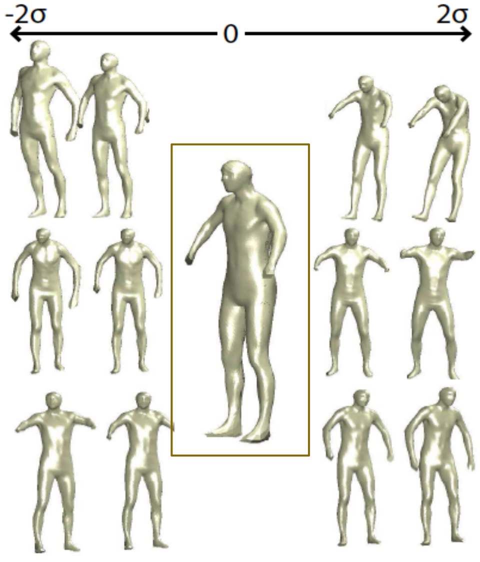
Only suitable for manifold shapes

- Manifold shapes are simple
  - Can be easily parameterized
  - There are elegant tools from differential geometry
  - They bend and stretch but they have a fixed topology

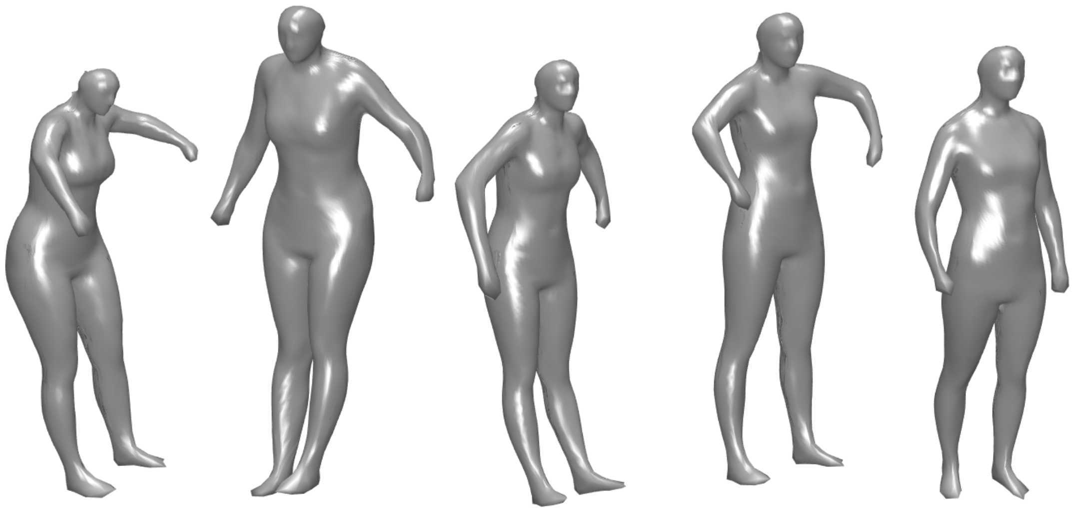
Laga et al. (2017) Numerical Inversion of SRNF maps for Elastic Shape Analysis. In IEEE PAMI



(a) Shape.



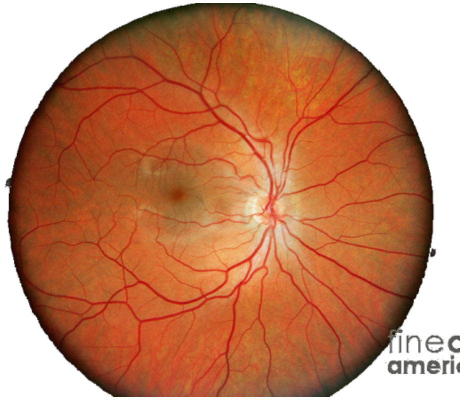
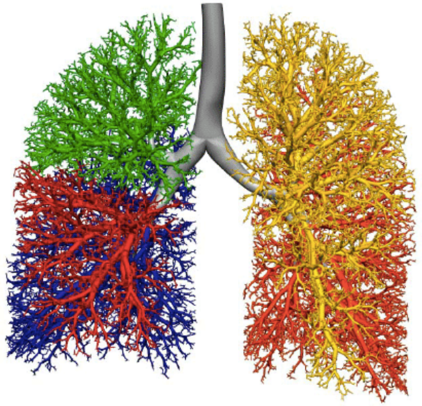
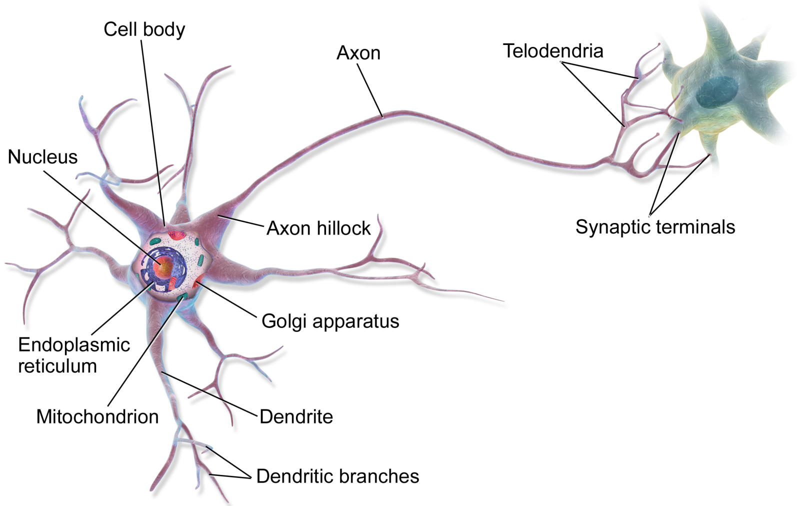
(b) Pose.

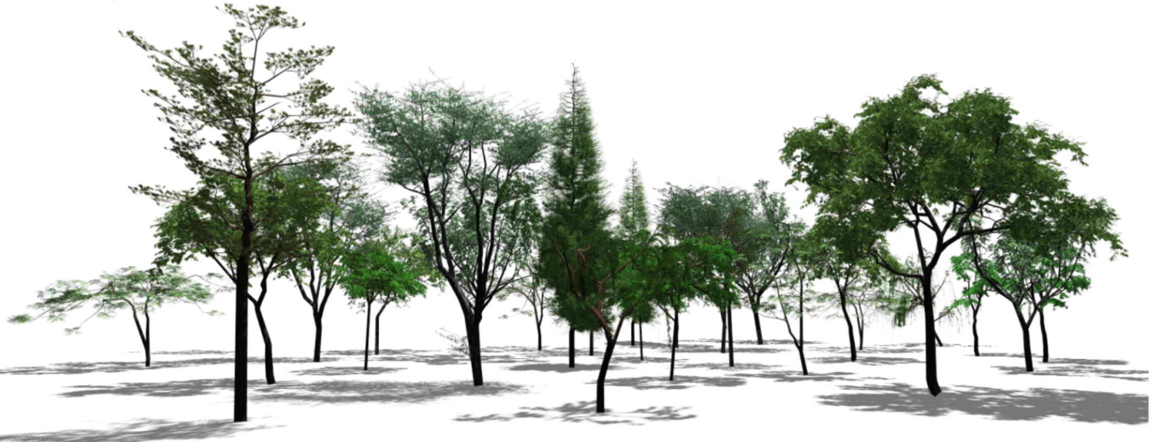


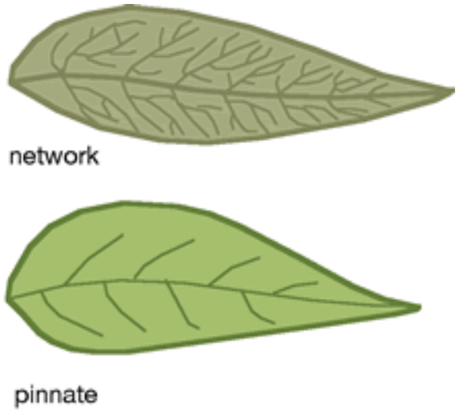
(c) Randomly synthesized 3D human bodies

What about objects that vary in topology?

- Many natural objects deform in geometry and topology





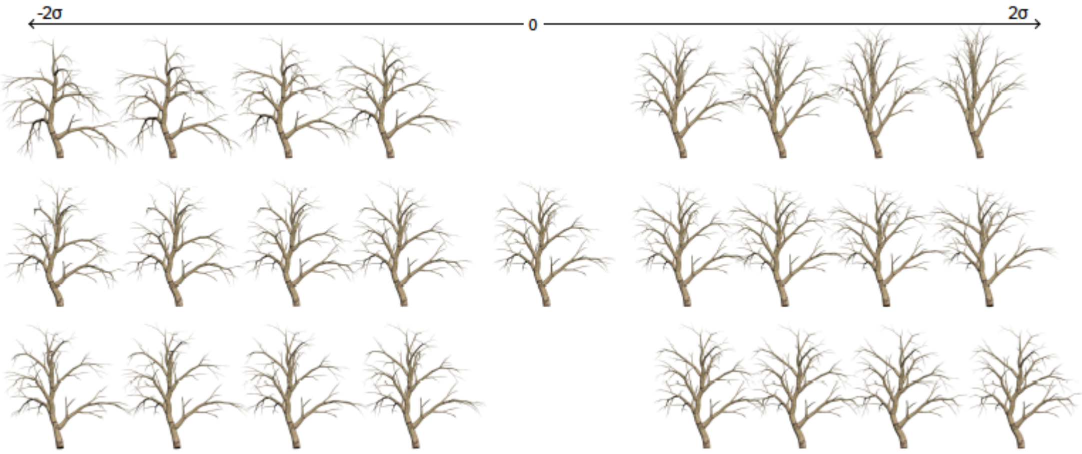
In this presentation

- Objects that have a tree structure

Correspondences and geodesics

Symmetry analysis

(a) Input trees.

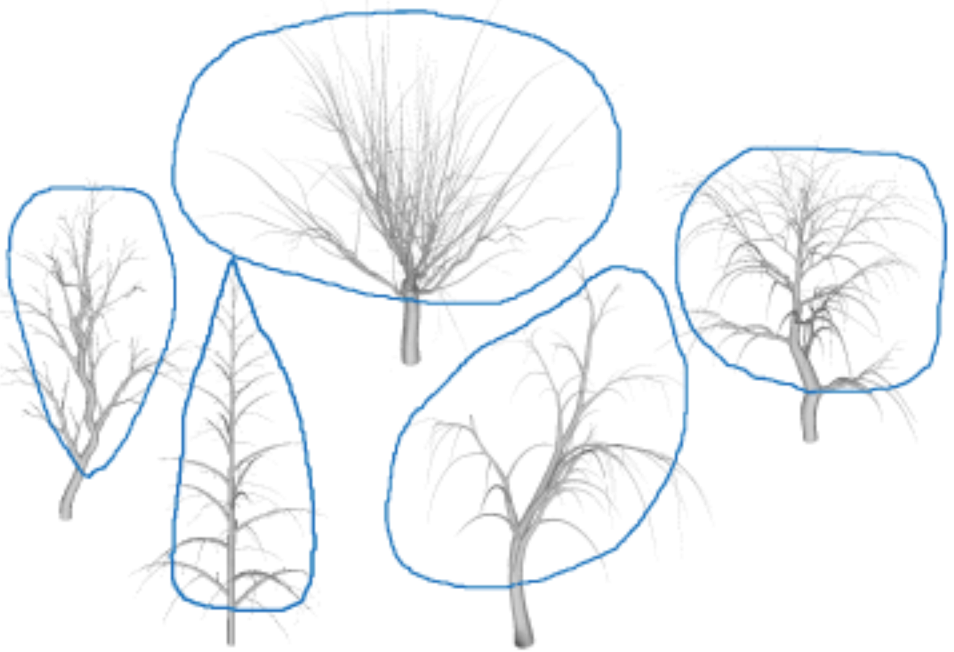


(b) Mean and first three modes of variation.

In this presentation

- Application to graphics (and biology)

Automatics synthesis of 3D botanical trees



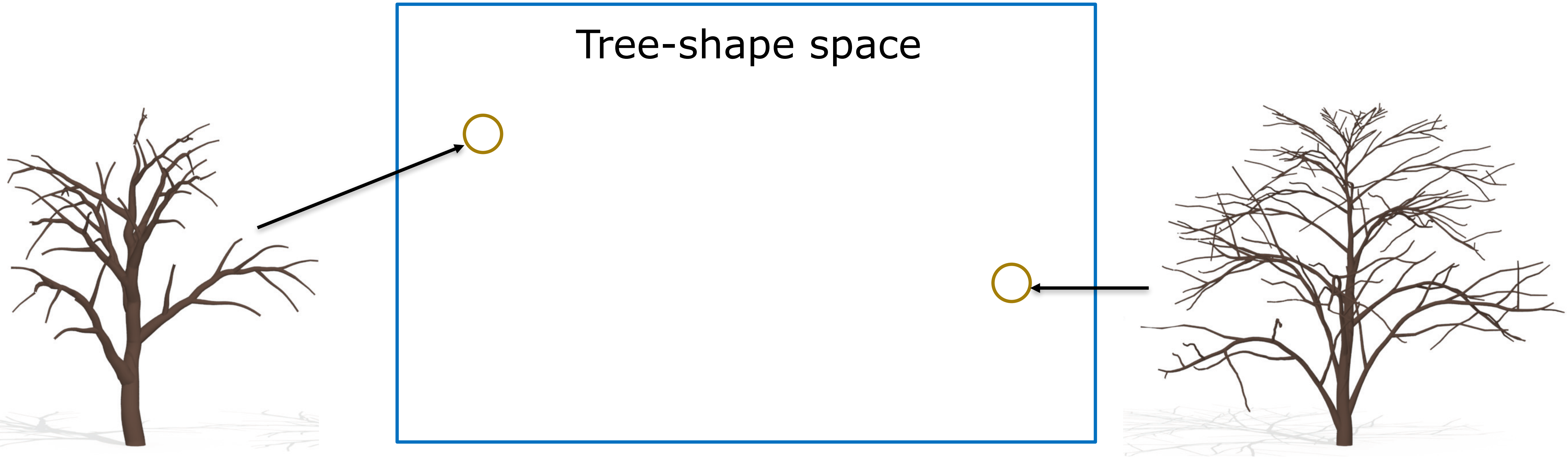
Sketch-based synthesis of 3D botanical trees

In this presentation

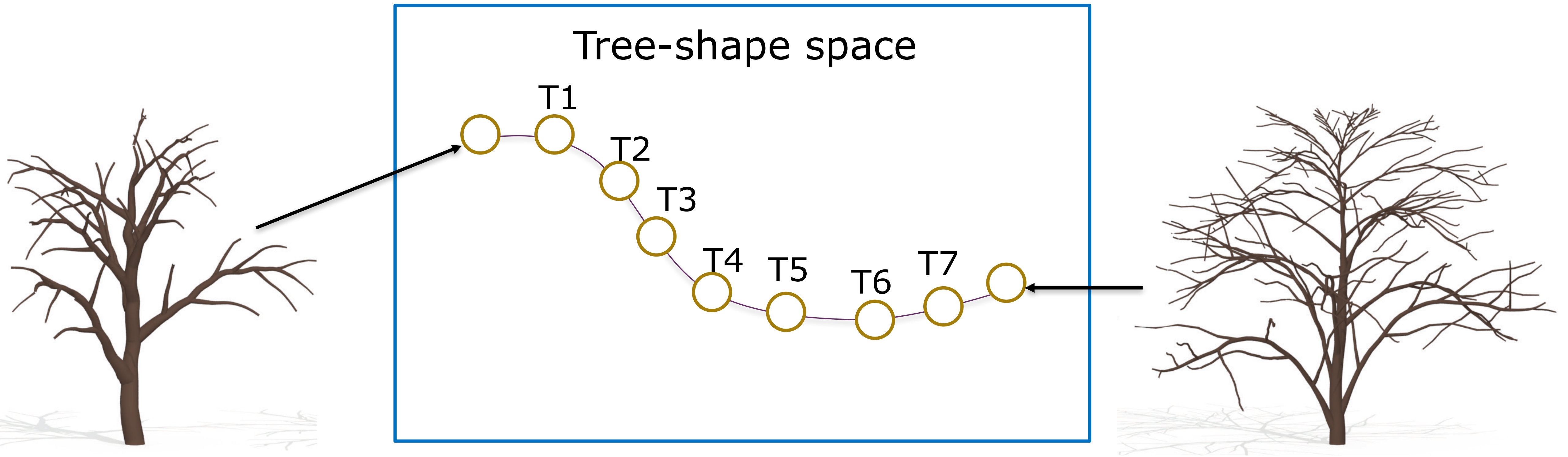
- Building blocks
  - A representation and a metric for measuring dissimilarities
    - It should measure geometric and topological deformations that change shape
  - Correspondences and geodesics
    - Geodesic is the shortest path, with respect to a metric, between two points
    - In shape analysis, it is the optimal path that deforms one shape onto another
- Applications in graphics and biology
  - Means and modes of variation for botanical trees and plant roots
  - Synthesizing botanical trees from a few parameters (regression)
  - Symmetry analysis

The general framework

The general framework

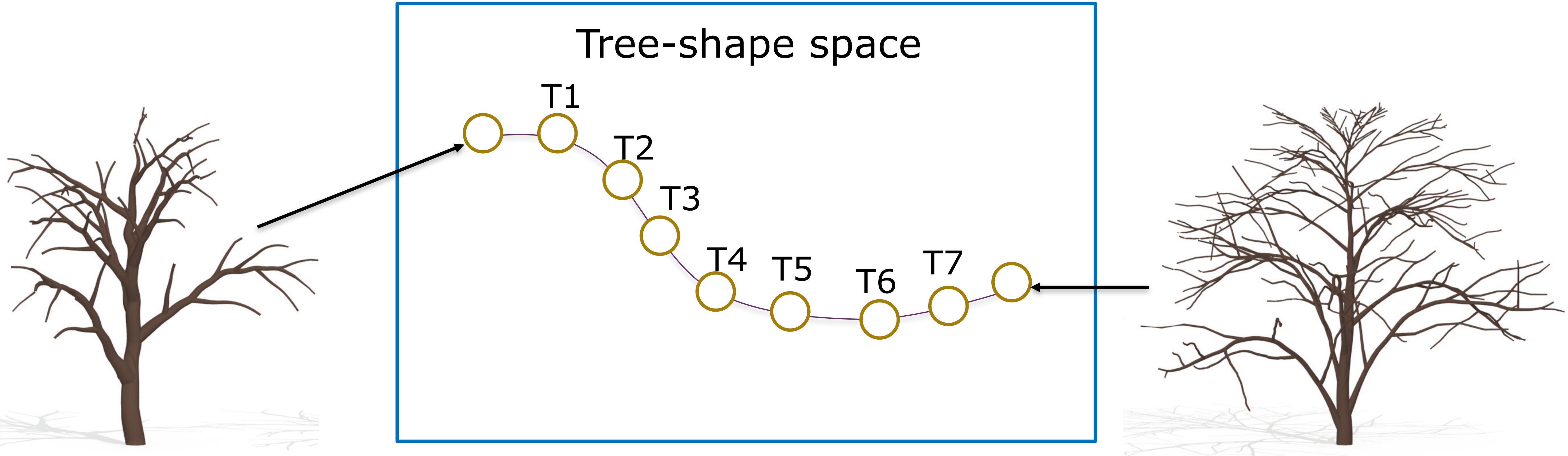


The general framework



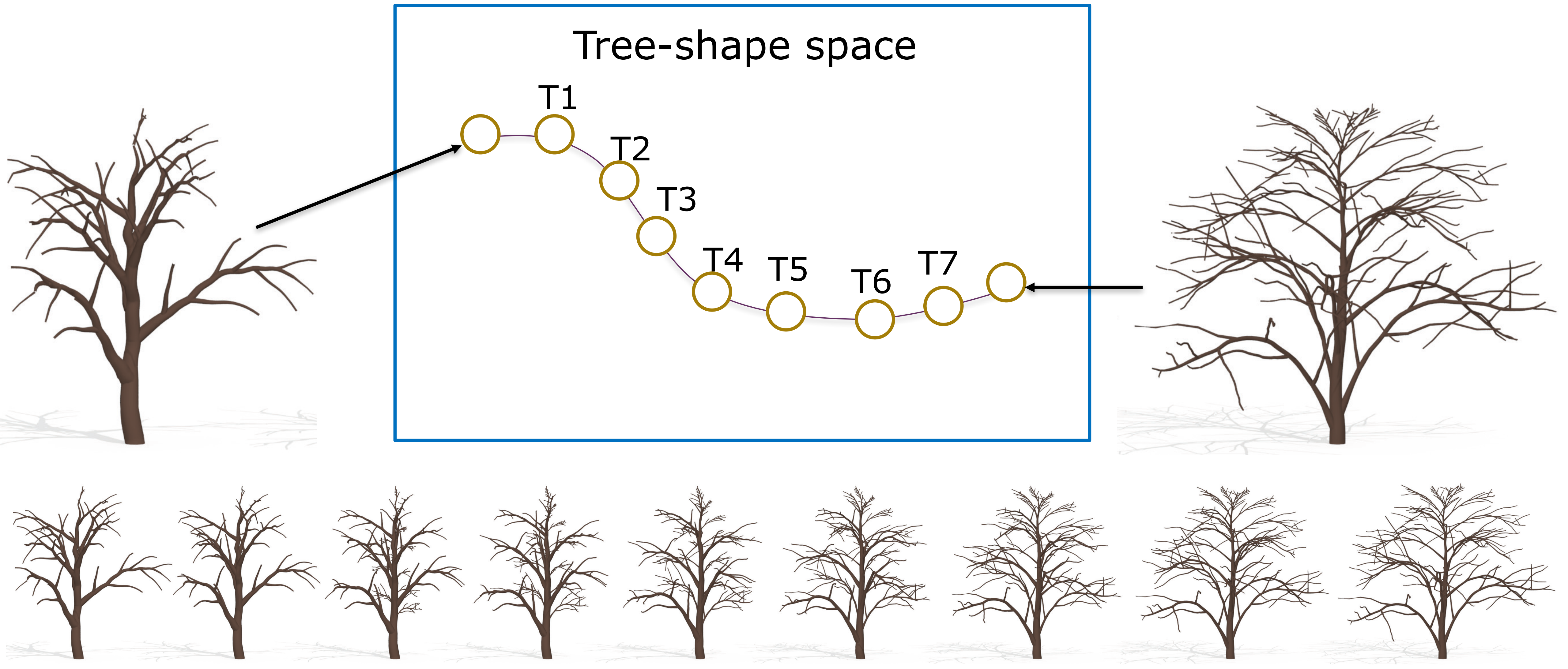
A path in the tree-shape space is a sequence of deformations (bending and stretching of branches, and changes in the tree topology)

The general framework

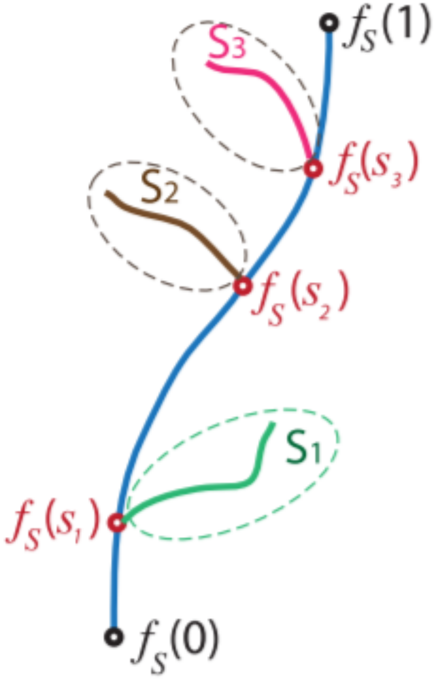


The shortest path, under the metric that quantifies bending, stretching, and topological changes, is called a geodesic

The general framework



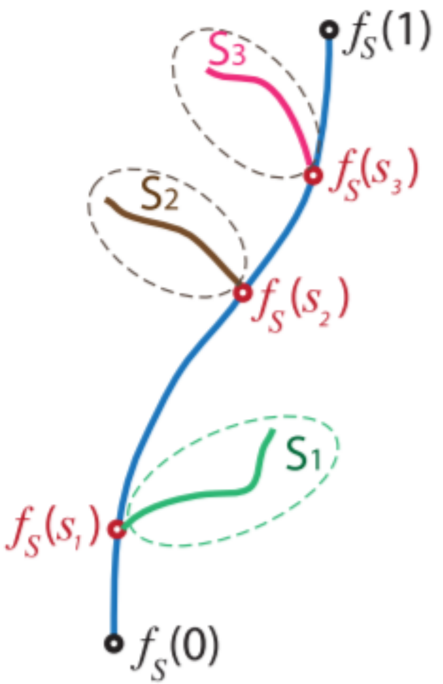
Representation of tree-like shapes



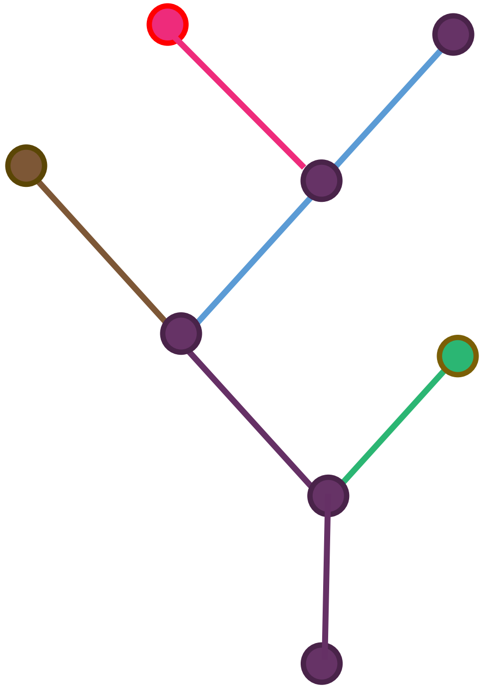
Input tree

Representation of tree-like shapes

- Tree-like shapes as a tree graph
  - Nodes $v \in V$ are bifurcation points, Edges $e \in E$ connect bifurcation points



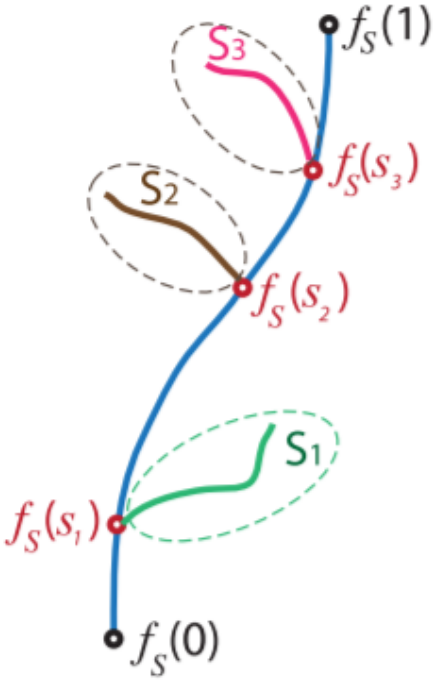
Input tree



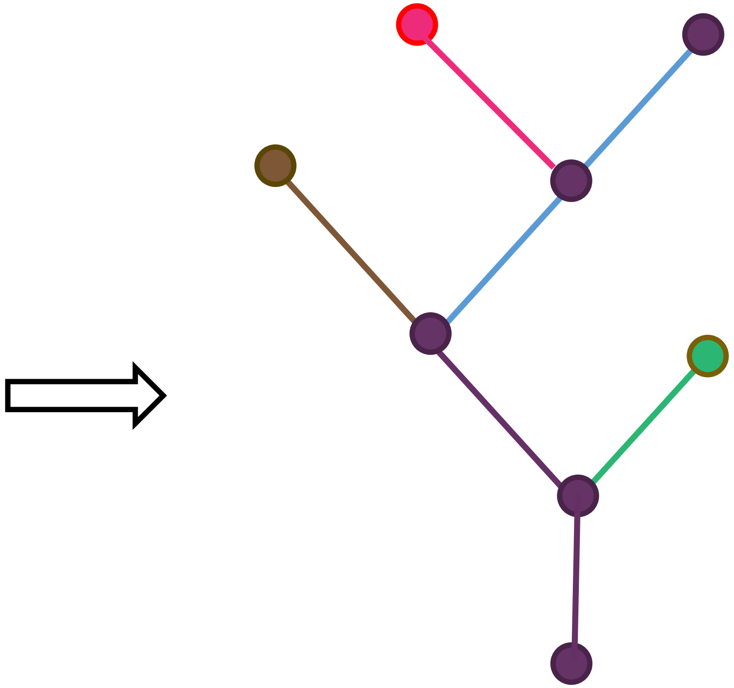
Tree structure

Representation of tree-like shapes

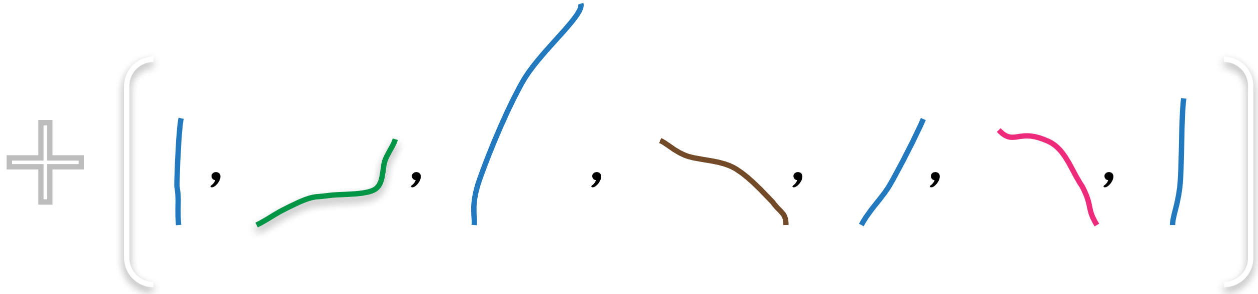
- Each edge has a geometry
  - 1D skeletal curve + thickness of the branch at each point



Input tree



Tree structure



Geometry of the edges

Representation of tree-like shapes

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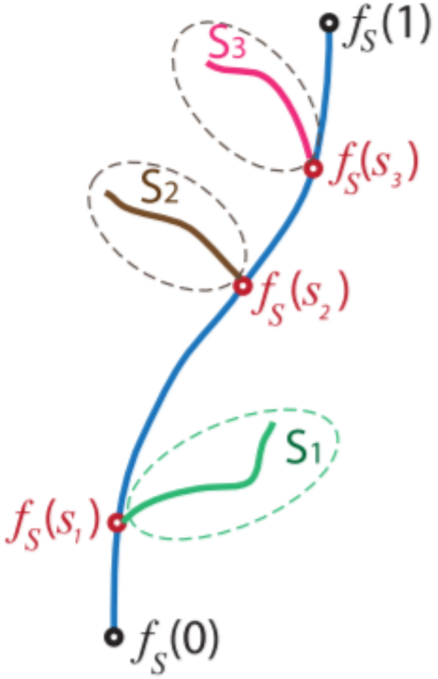
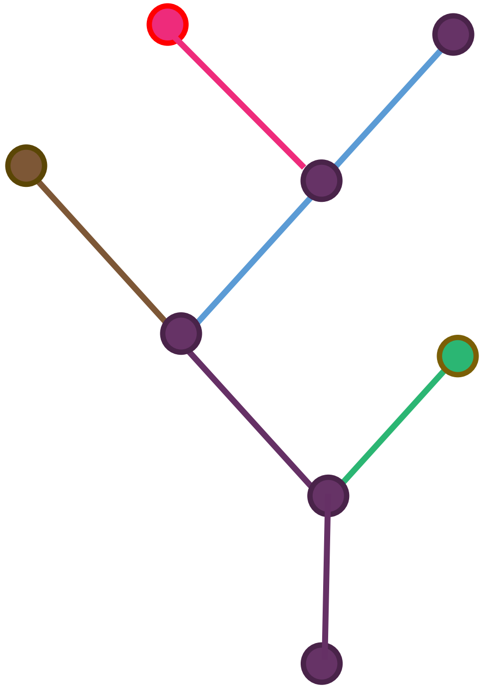
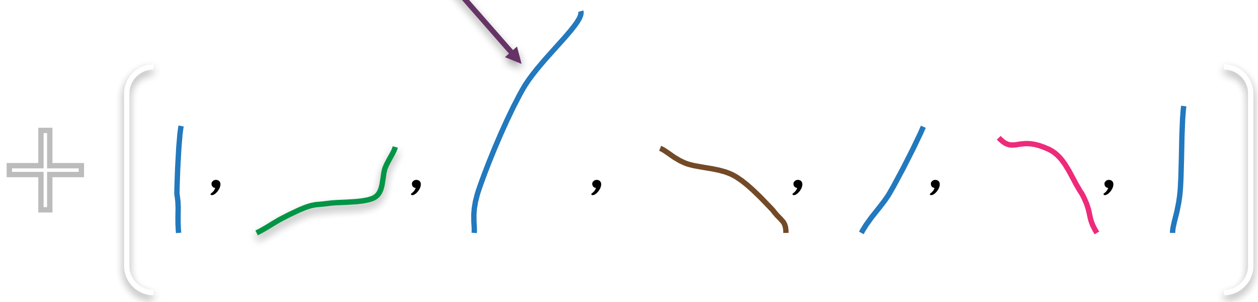


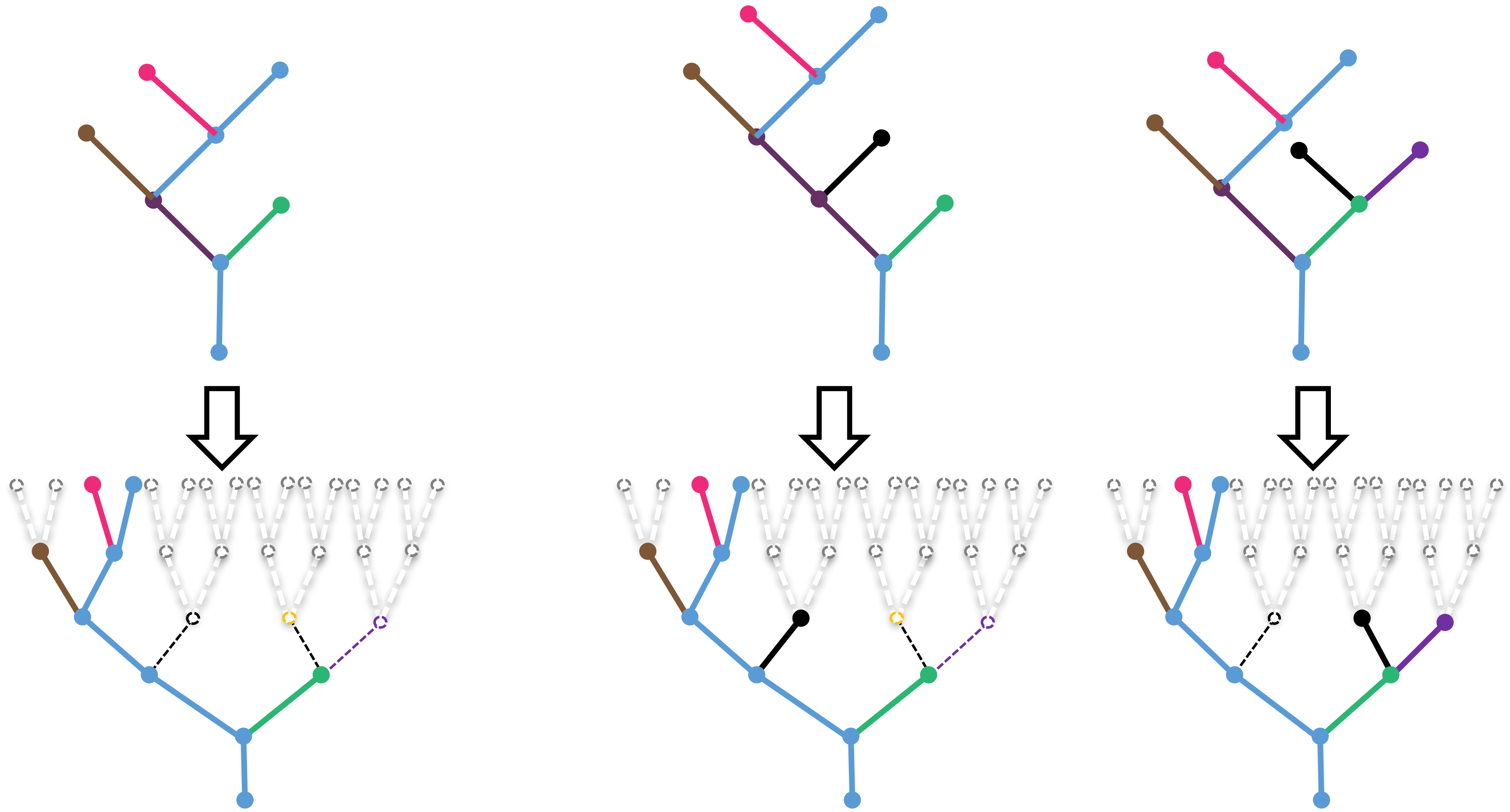
Input tree


Tree structure

$x_e \in F_e = \left(\mathbb{R}^d\right)^n, d = 4,$
n is the no. of sample points

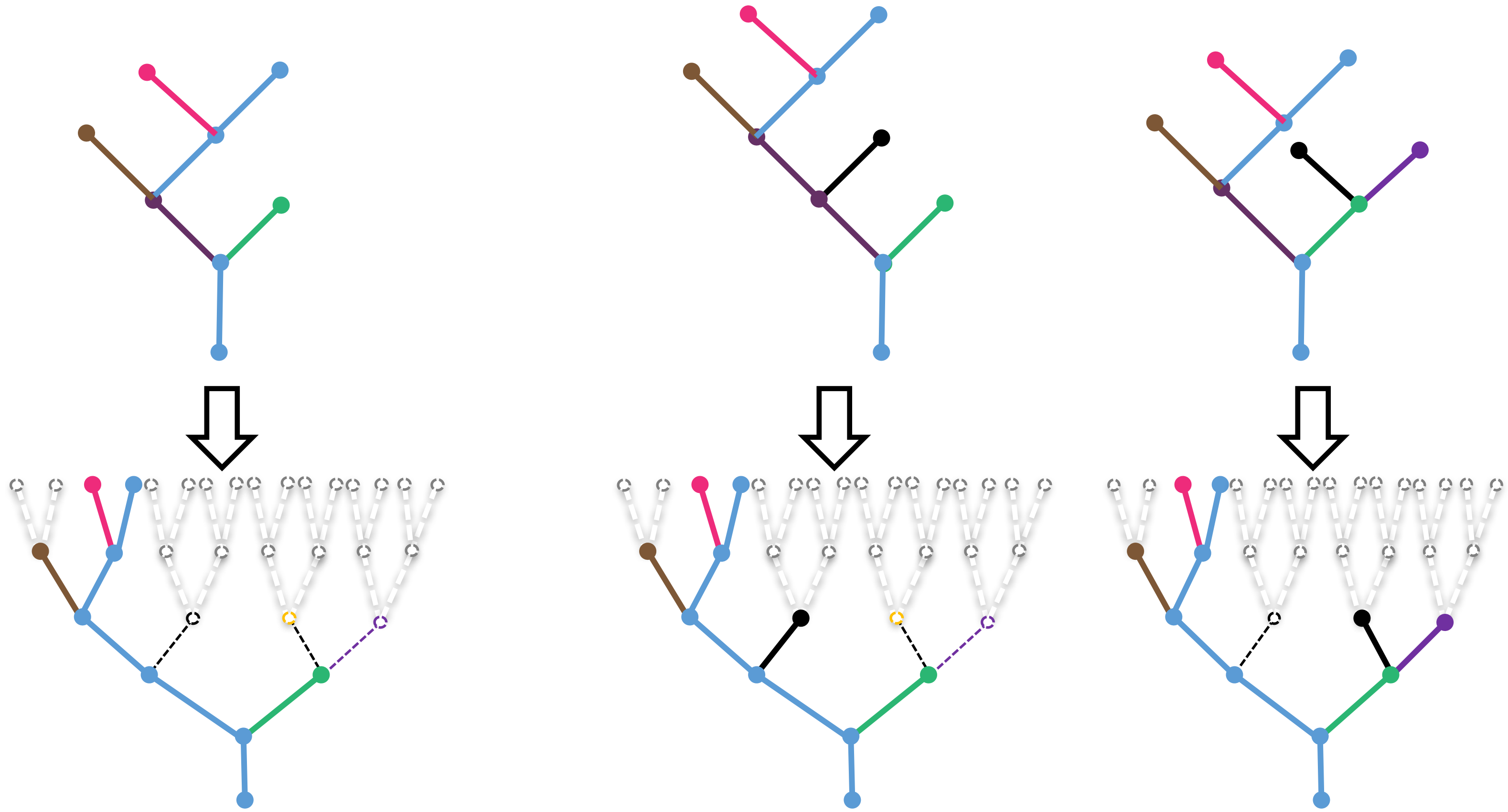

Geometry of the edges

Parameterization with maximal binary trees



Dotted edges are virtual (collapse) edges of length

Parameterization with maximal binary trees



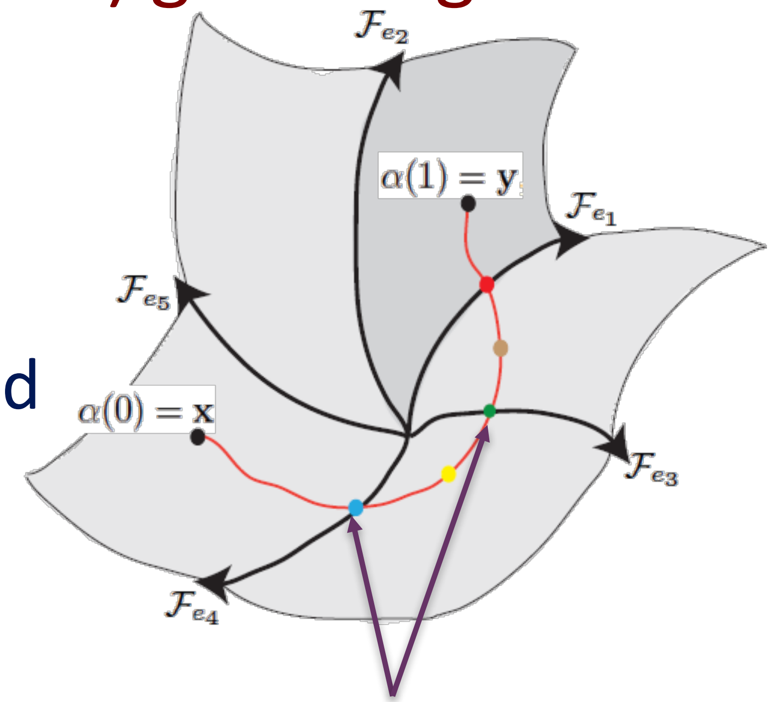
A tree becomes an element of $X = \mathcal{F}_{e_1} \times \cdots \times \mathcal{F}_{e_m}$,

The tree-shape space and the metric

- A tree is an element of

$$X = \mathcal{F}_{e_1} \times \cdots \times \mathcal{F}_{e_m};$$

- The tree-shape space is set of subspaces (orthants) glued together
  - A subspace $X_k = \mathcal{F}_{e_{i_1}} \times \cdots \times \mathcal{F}_{e_{i_k}}$. contains trees whose non-zero edges are $(e_1, \cdots, e_k)$
- Within the same orthant
  - Geometry varies but structure remains unchanged
- Transitions across orthants
  - changes in topology (edge collapse, node split)



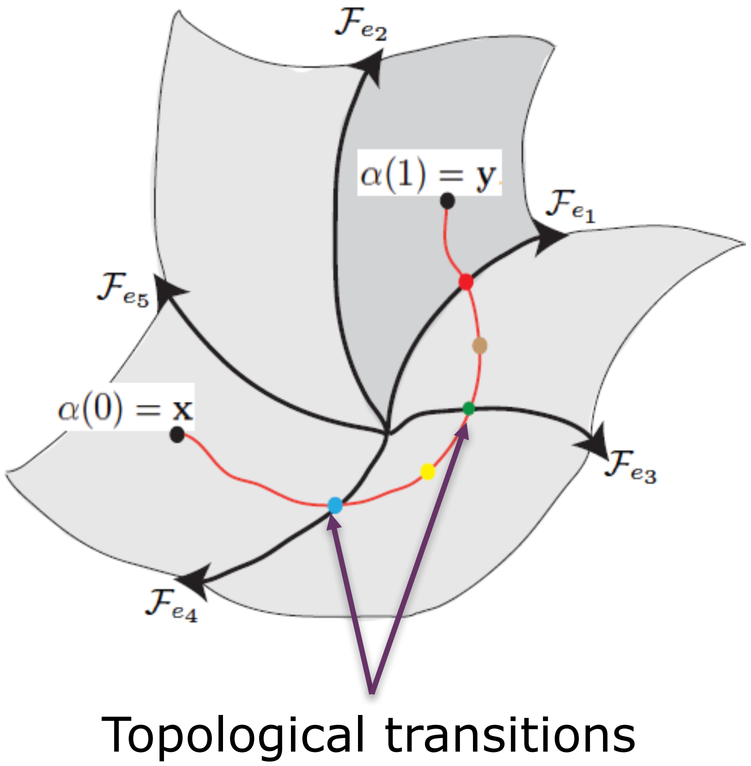
Topological transitions

Metrics and geodesics

- A geodesic is the shortest path between two trees x and y
  - It can go through multiple orthants (edge collapses, node splits)
  - It is the solution to

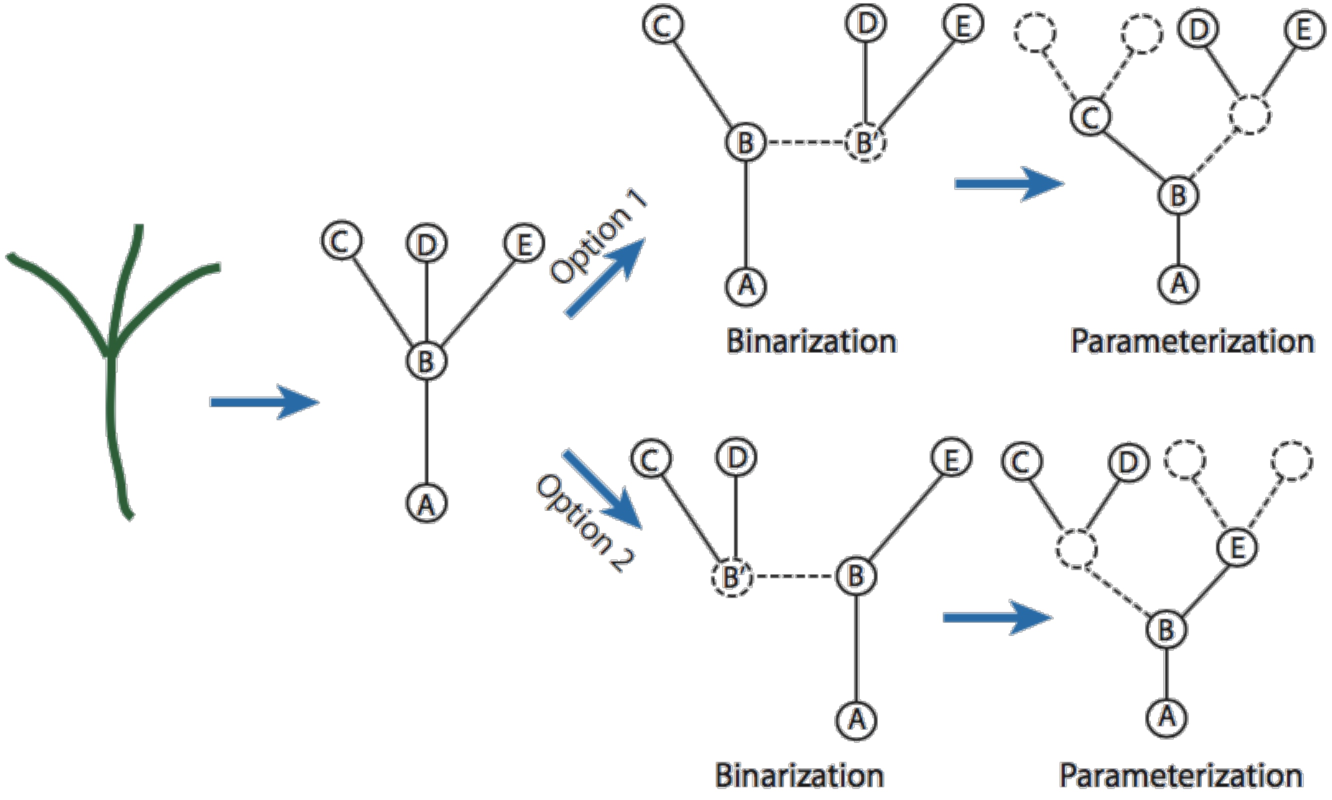
$$\arg\min \left\{ d_g^2(x, y) = \sum_{t=1}^{N-1} d^2(x_t, x_{t+1}) \right\}$$

- The minimization is over all possible rigid transformations and re-parameterizations of the two trees



The parameterization is not unique

- Binary trees that collapse to the same tree-shape are equivalent



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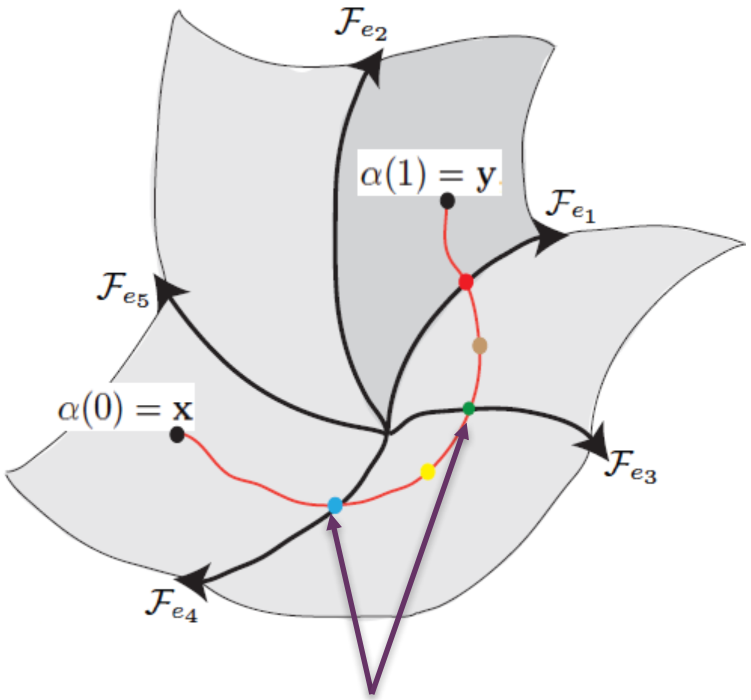
- The quality of the geodesic depends on the choice of the metric $d(.\,,.)$



Topological transitions

A metric which quantifies bending and stretching

- Bending corresponds to changes in
  - The orientation of the tangent vectors to the skeletal curve
- Stretching corresponds to
  - Branch elongation, which can be measured by changes in the magnitude of the tangent vector to the skeletal curve
  - Changes in the thickness of the branches

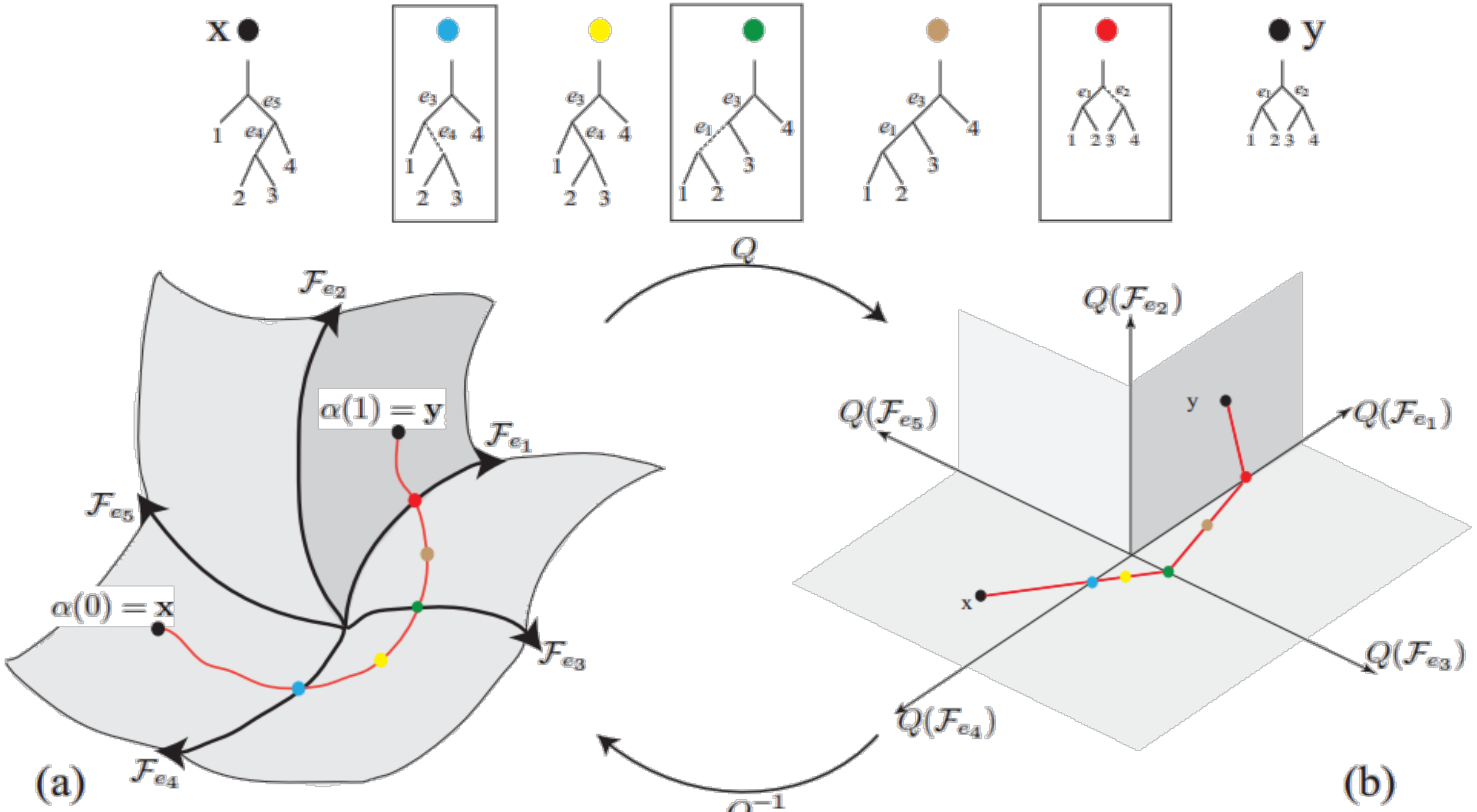
$$\langle\langle\dot{\beta},\dot{\beta}\rangle\rangle \equiv a\int \underbrace{\langle\theta'(s),\theta'(s)\rangle e^{\phi(s)}ds}_{\text{Bending}} + b\int \underbrace{\phi'(s)^2 e^{\phi(s)}ds}_{\text{Elongation}} + c\int \underbrace{r'(s)^2 e^{\phi(s)}ds}_{\text{Change in thickness}}.$$

The ESRVF tree-shape space

- If we set
  - $a = c = 1$, $b = \frac{1}{4}$
- and define a branch with its Extended Square-Root Velocity function (ESRVF) (q, r), where
  - q is the square-root velocity function (SRVF) of the skeletal curve $$q(s) = \frac{f'(s)}{\|f'(s)\|^{\frac{1}{2}}}.$$
  - r is the thickness
- The complex metric becomes an L2 metric in the ESRVF space

The ESRVF tree-shape space

- Map all trees onto the ESRVF space, use the QED metric to compute geodesics, and map the result back for visualization



Tree-shape space

ESRVF Tree-shape space

Examples of geodesics

QED in tree-shape space
[Wang et al. ACM ToG2018]

QED in the ESRVF tree-shape space [Wang et al. SGP2018]

Examples of geodesics

QED in tree-shape space [Wang et al. ToG2018]

QED in the ESRVF tree-shape space [Wang et al. SGP2018]

Examples of geodesics (video 1 and video 2)

QEDT in tree-shape space
[Wang et al. ToG2018

QEDT in the ESRVF
tree-shape space
[This Article]

QEDT in tree-shape space
[Wang et al. ToG2018]

QEDT in the ESRVF
tree-shape space
[This Article]

Application to symmetry analysis

Shape f

Application to symmetry analysis

Shape *f*

$$\tilde{f} = H(v) f$$

(Reflection of *f* with respect to an arbitrary plane)

$$H(v) = \left(I - 2\frac{vv^T}{v^T v}\right)$$

Application to symmetry analysis

Shape *f*

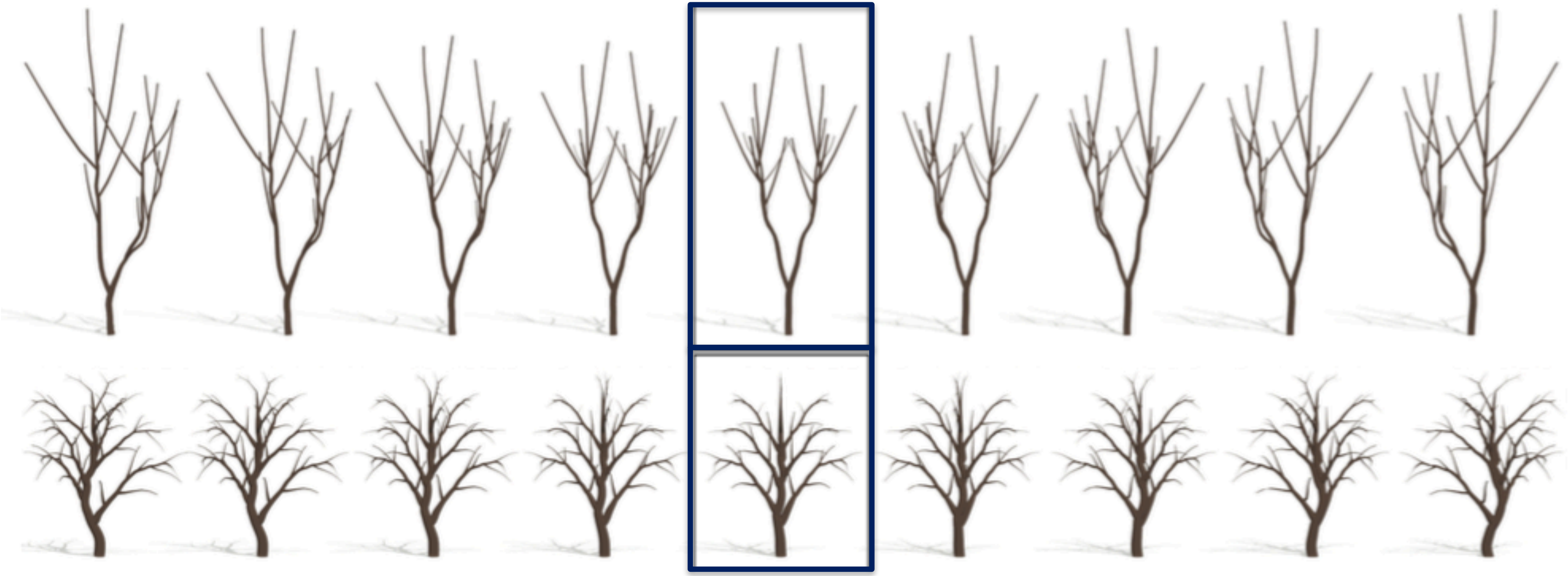
Fully symmetric tree

$$\tilde{f} = H(v)f$$

(Reflection of *f* with respect to an arbitrary plane)

$$H(v) = \left(I - 2\frac{vv^T}{v^T v}\right)$$

Application to symmetry analysis



Fully symmetric tree

Applications – Summary statistics

- Mean tree of a population of trees
  - Map all the trees to the ESRVF space
  - Compute their mean by solving

$$\mu_q = \arg\min_{\mathbf{q}} \sum_{i=1}^{n} [d(\mathbf{q}, \mathbf{q}_i)]^2,$$

  - Map the result back to the tree-shape space for visualization

Applications – Summary statistics

- Modes of variation (Geodesic PCA)
  - Map all the trees to the ESRVF space
  - Compute their mean
  - Project all points to the tangent space at the mean
  - Perform PCA in the tangent space
  - Map the principal directions to principal geodesics in the ESRVF space
  - Map the principal geodesics in the ESRVF space to the tree-shape space for visualization

Applications – Random sampling

- Synthesizing random trees
  - Perform PCA in the tangent space to the ESRVF space
  - Generate random samples in this tangent space

$$\left(\sum_{i=1}^{N} \alpha_i \sqrt{\lambda_i} \Lambda_i\right)$$

  - Map the sample to the ESRVF space

$$\mathbf{q} = \mathrm{Exp}_{\mu_q}\left(\sum_{i=1}^{N} \alpha_i \sqrt{\lambda_i} \Lambda_i\right)$$

  - Map the sample back to the trees-shape space for visualization

Application - Mean tree computation

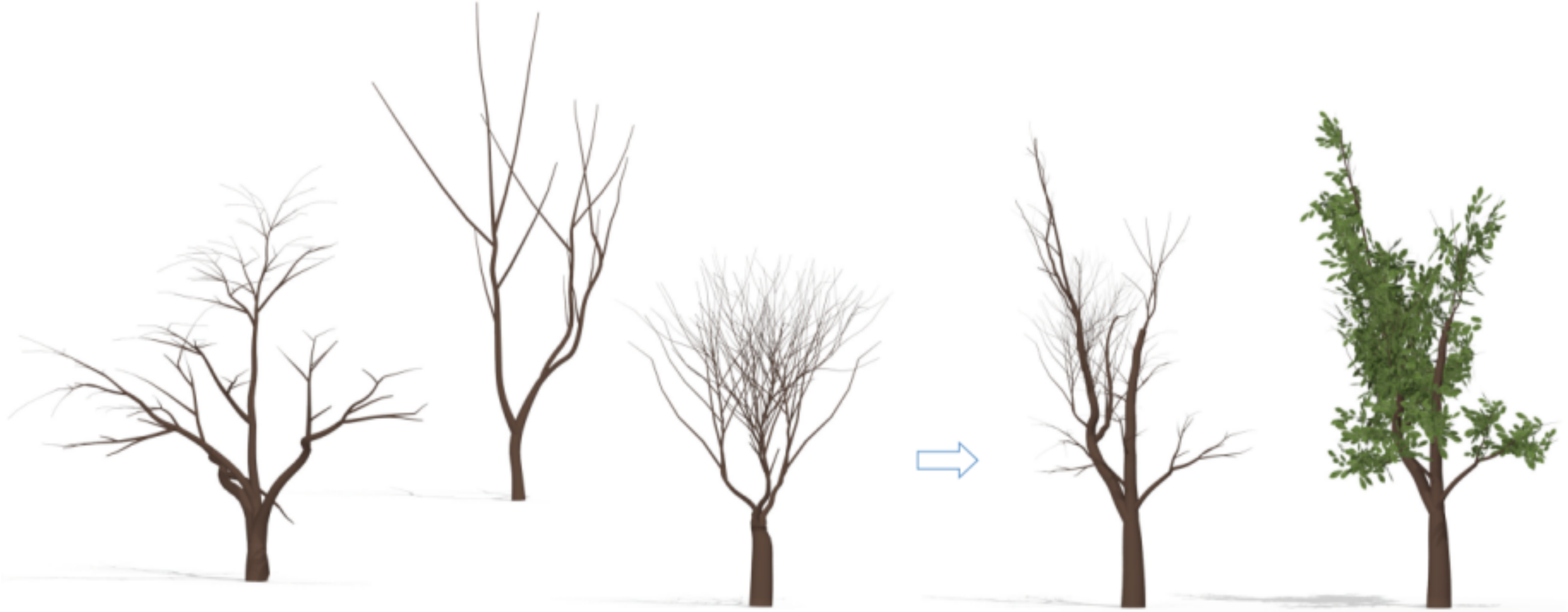
Input tree models $\{x_i, i = 1, \cdots, N\}$

Application - Mean tree computation

Input tree models $\{x_i, i = 1, \cdots, N\}$

$$\mu = \arg\min_{x \in X} \sum_{i=1}^{N} d_g^2(x, x_i)$$

Examples of mean trees



Input tree models

Mean tree

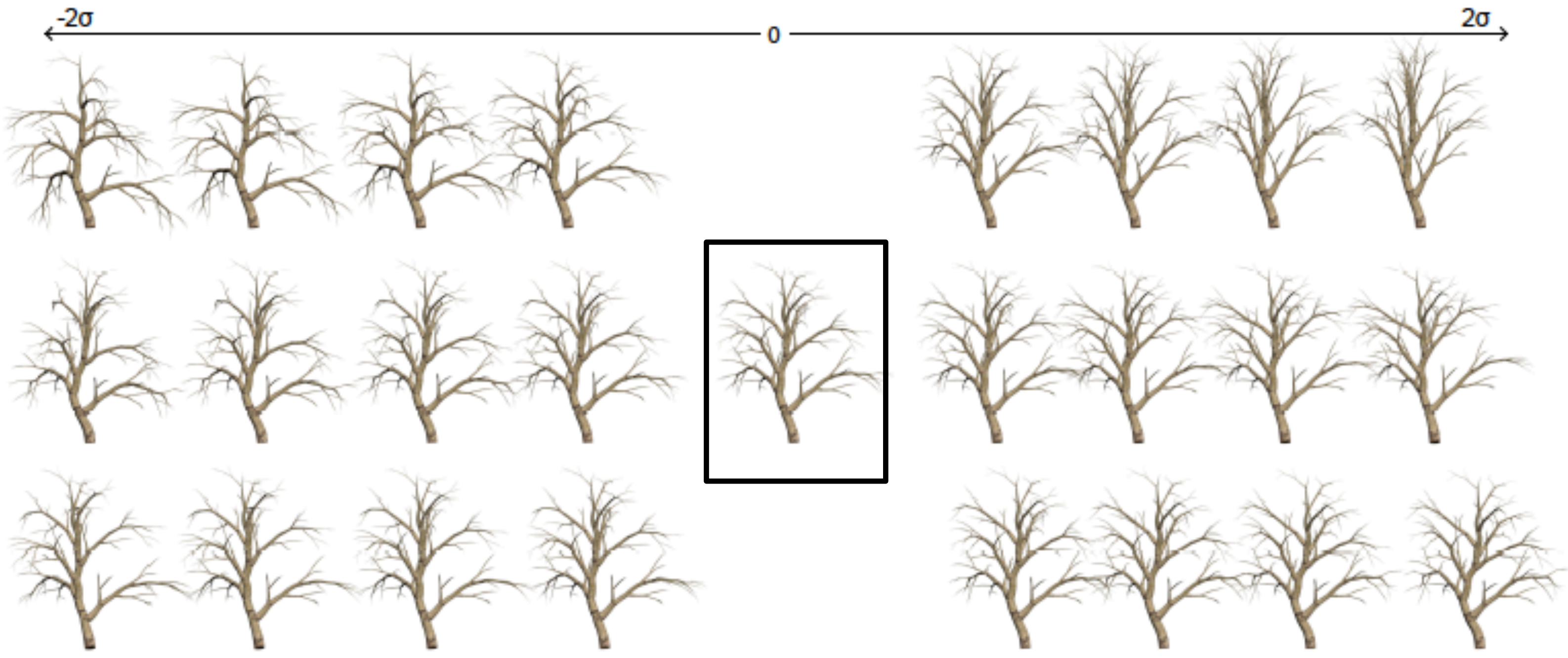
Examples of mean trees

Input tree models

Mean tree

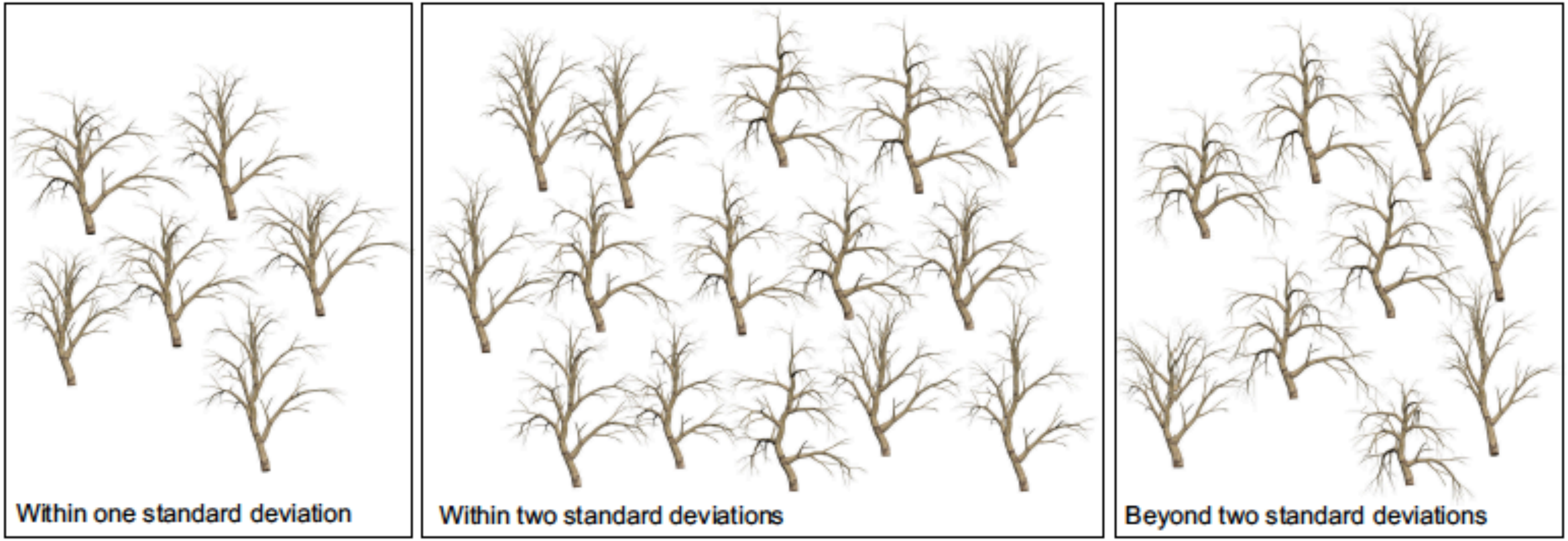
Applications – Summary statistics

(a) Input trees.



(b) Mean and first three modes of variation.

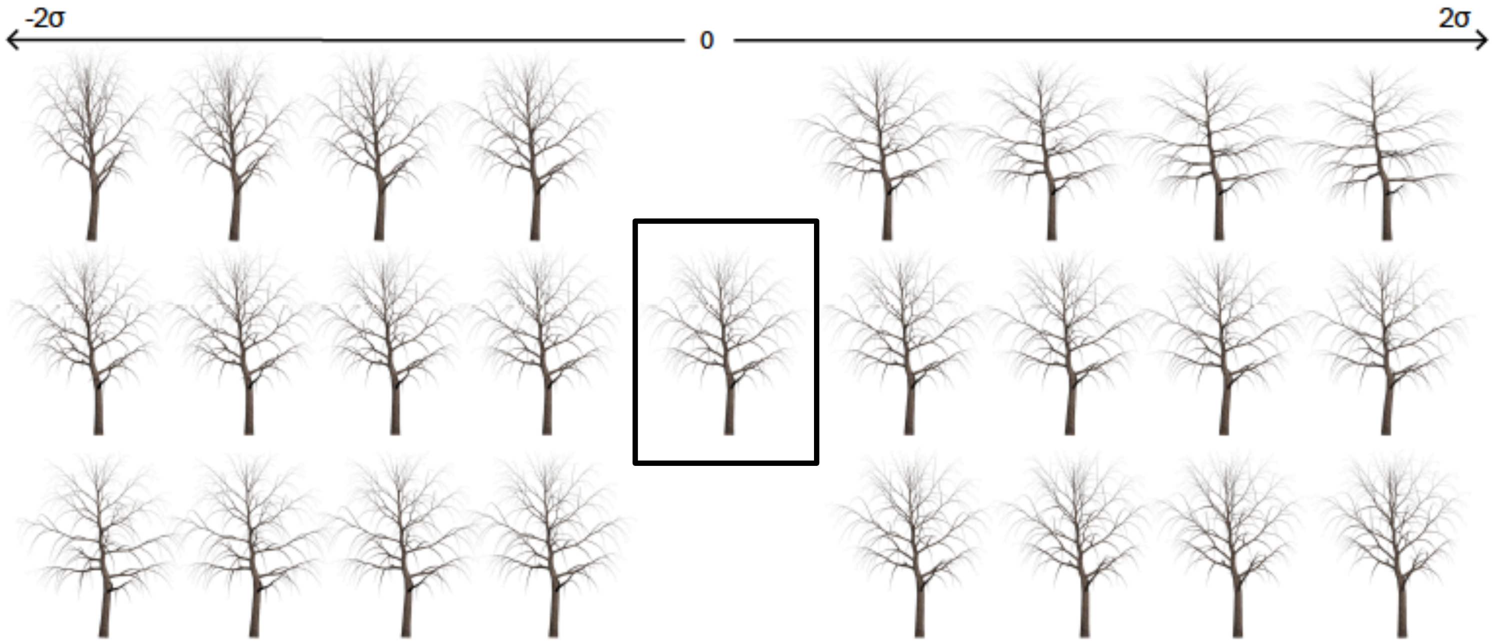
Applications – Summary statistics



Automatically synthesized random trees

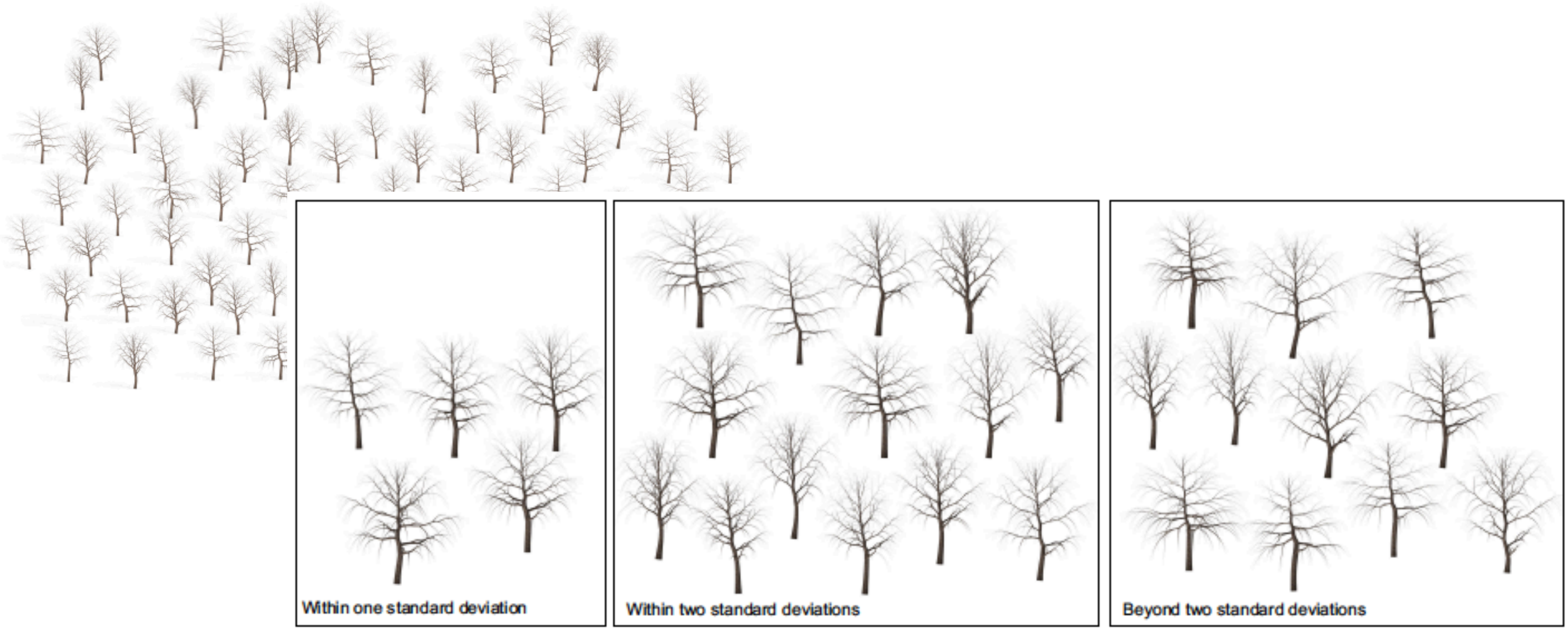
Applications – Summary statistics

(a) Input trees.



(b) Mean and first three modes of variation.

Applications – Examples of summary statistics



Automatically synthesized random trees

Applications – Regression

- Any tree can be written in the form $\mathbf{q} = \mathrm{Exp}_{\mu_q}\left(\sum_{i=1}^{N} b_i \cdot \Lambda_i\right)$
- Let p be a set of m parameters
  - Assume that any tree can be obtained from these parameters using the linear relation

$$A \times \mathbf{p} = \mathbf{b}$$

  - **A** is the regression matrix
  - **b** is a vector which holds the b_i's

Applications – Regression

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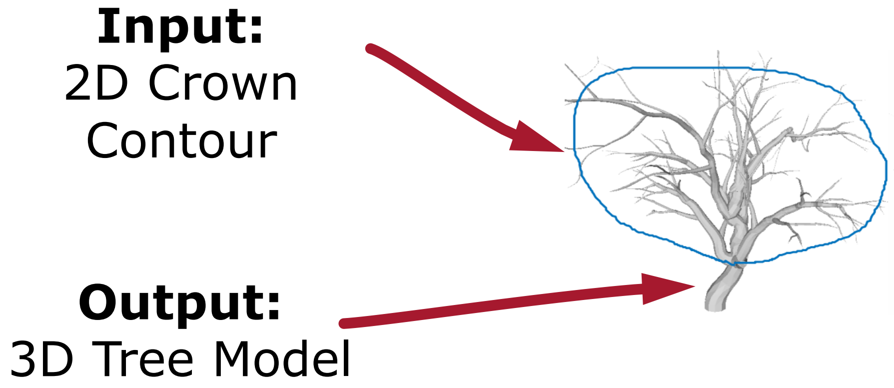
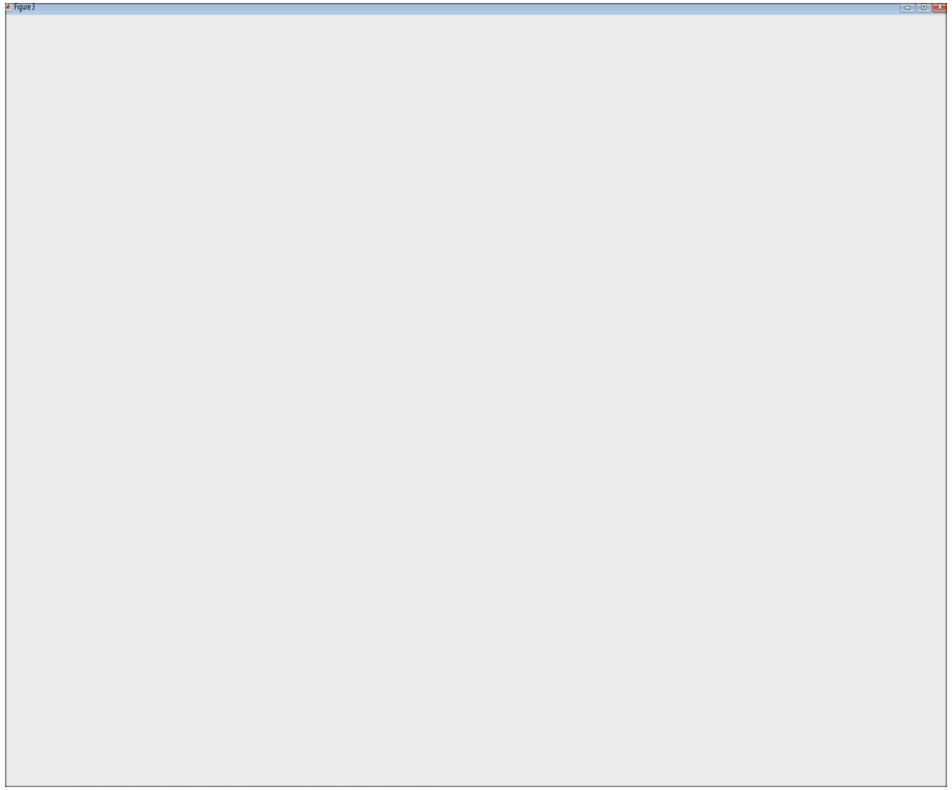
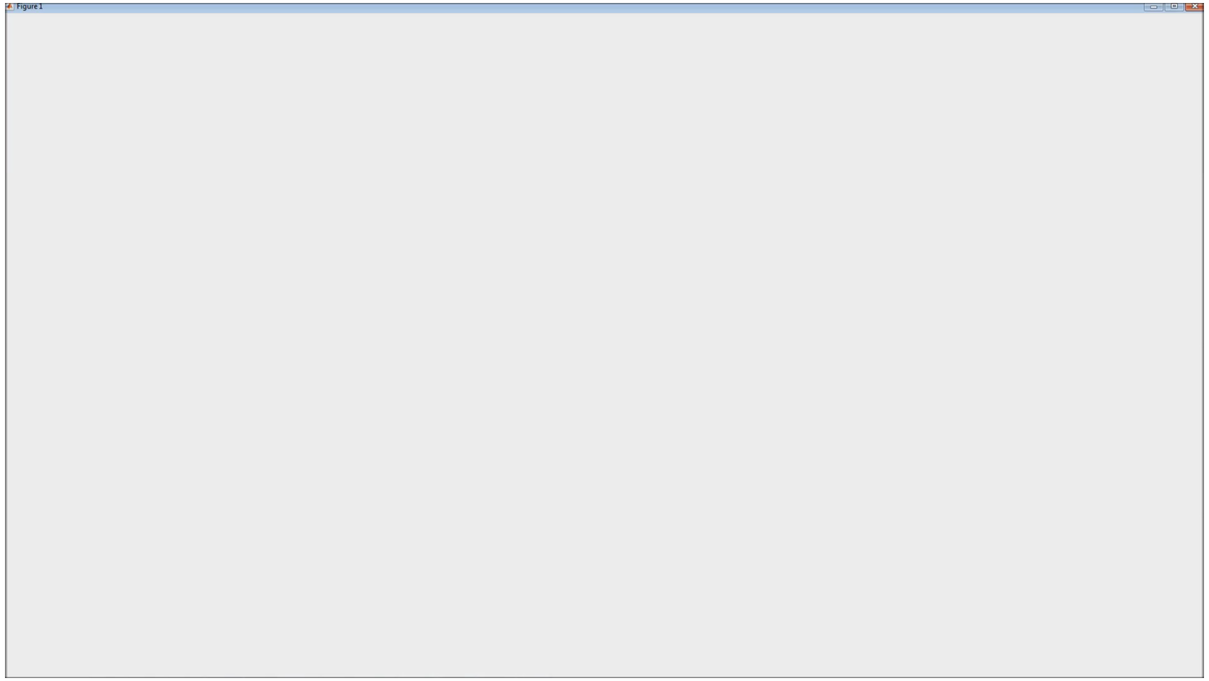
- The parameters **b** can be
  - Biologically motivated
  - 2D sketches in our case



Applications – Regression (video 3 and video 4)

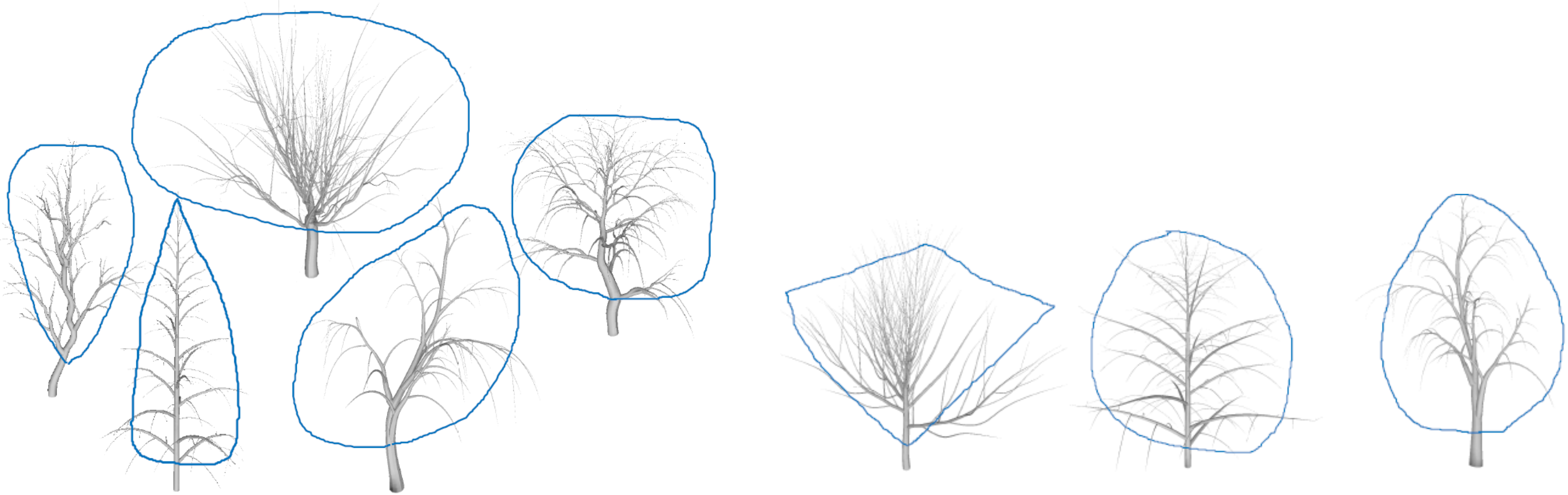
- Sketch-based 3D tree synthesis (videos)



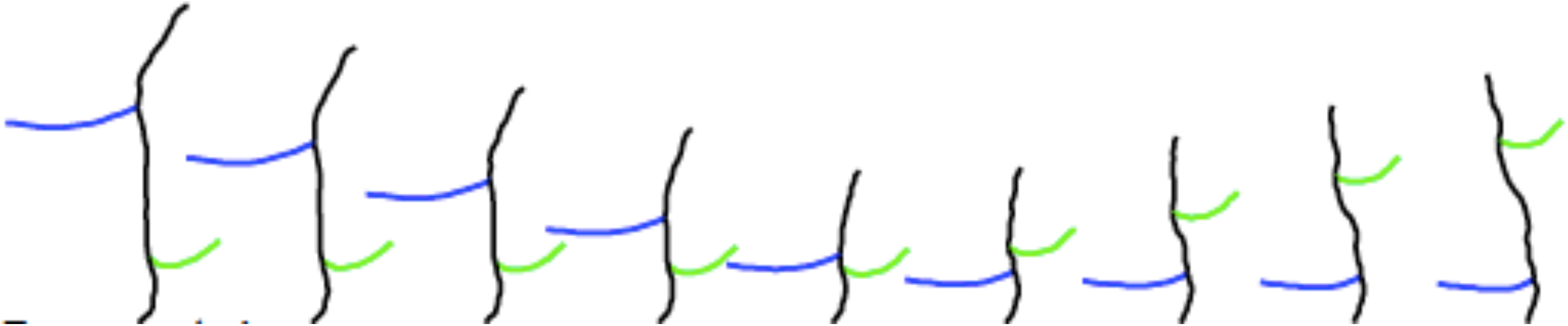


Applications – Regression

- Sketch-based 3D tree synthesis



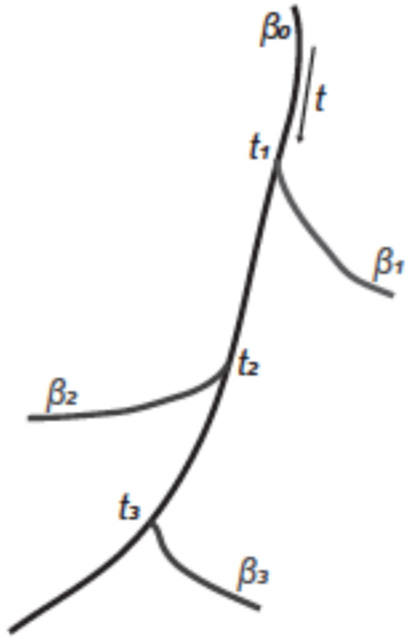
Limitation



Formulation 2

- A tree as layers of curves
  - Each side branch will grow at a location of the main parent branch
  - Each curve will be represented with its SRVF function

$$q(s) = \frac{f'(s)}{||f'(s)||^{\frac{1}{2}}}.$$



Formulation 2

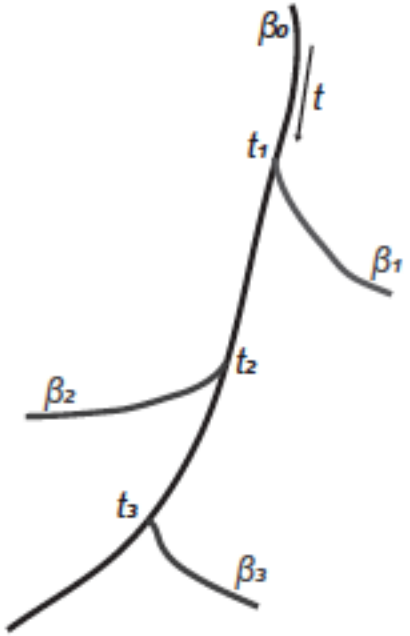
- In the case of two layers (one main branch and N lateral branches), each tree i will be represented as

$$\tilde{\boldsymbol{q}}^{\boldsymbol{i}} = (\tilde{q}_0^i, \{(\tilde{q}_k^i, \tilde{s}_k^i)\}_{k=1}^N)$$

SRVF of the main branch

SRVF of the i-th lateral branch

Location of the i-th
lateral branch on the main bracnh



Formulation 2

- The metric

$$d_{\mathcal{C}}(\tilde{\boldsymbol{q}}^{\mathbf{1}}, \tilde{\boldsymbol{q}}^{\mathbf{2}}) = \lambda_m \boxed{\| \tilde{q}_0^1 - \tilde{q}_0^2 \|^2} + \lambda_s \sum_{k=1}^{N} \boxed{\| \tilde{q}_k^1 - \tilde{q}_k^2 \|^2} + \lambda_p \sum_{k=1}^{N} \boxed{(\tilde{s}_k^1 - \tilde{s}_k^2)^2}$$

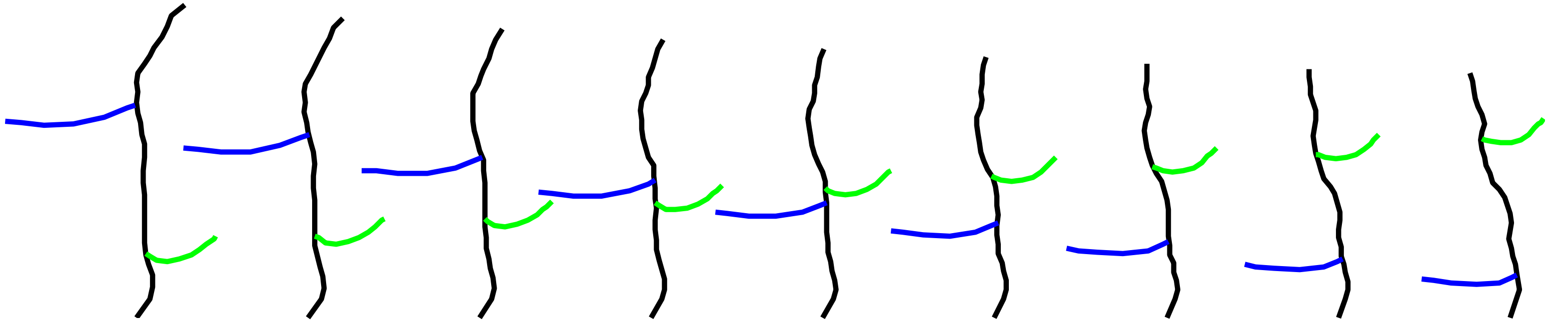
Formulation 2

- The metric
- The parameters $\boldsymbol{\lambda} = (\lambda_m, \lambda_s, \lambda_p)$ control the cost of
  - Deforming the main branch
  - Deforming the lateral branches
  - Moving the position of the lateral branches along their parent branch

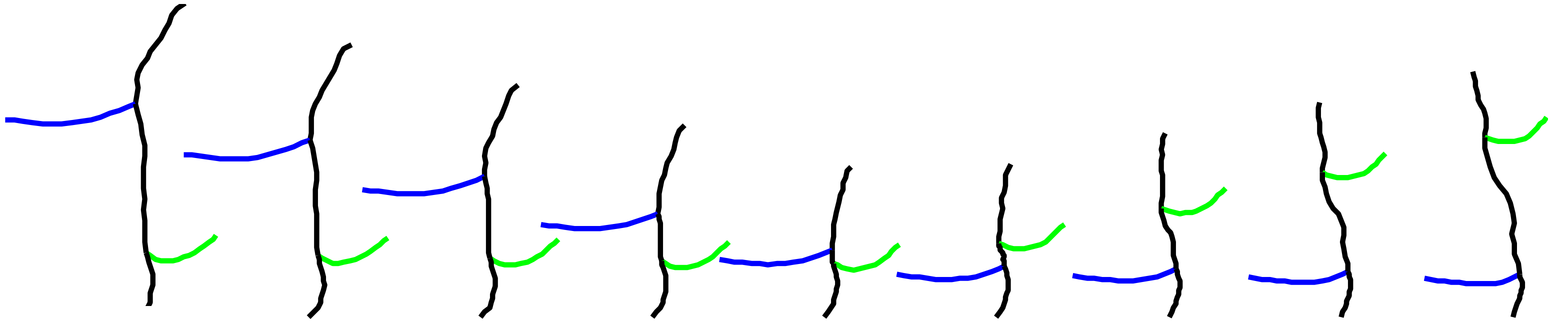
$$d_C(\tilde{\boldsymbol{q}}^1, \tilde{\boldsymbol{q}}^2) = \lambda_m \| \tilde{q}_0^1 - \tilde{q}_0^2 \|^2 + \lambda_s \sum_{k=1}^{N} \| \tilde{q}_k^1 - \tilde{q}_k^2 \|^2 + \lambda_p \sum_{k=1}^{N} (\tilde{s}_k^1 - \tilde{s}_k^2)^2$$

Some preliminary results

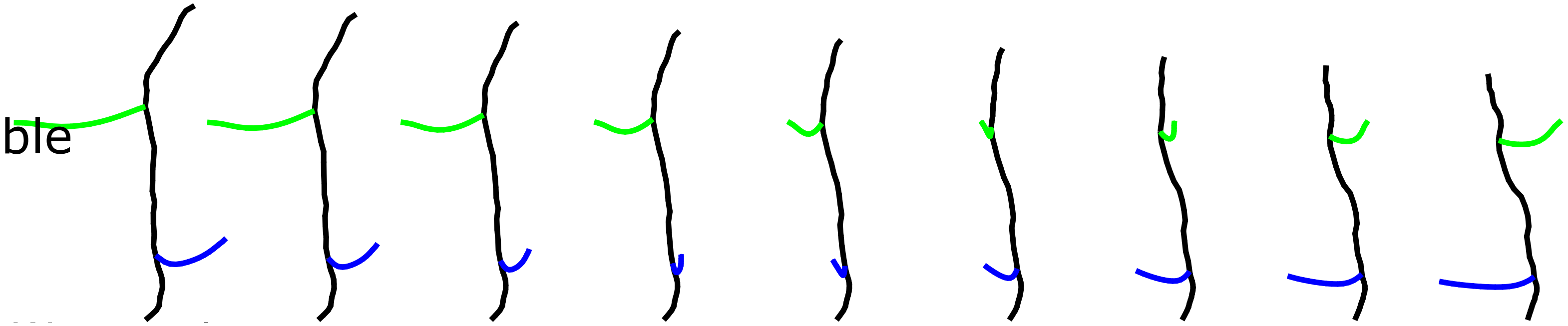
This approach



Previous approach with one possible branch correspondence

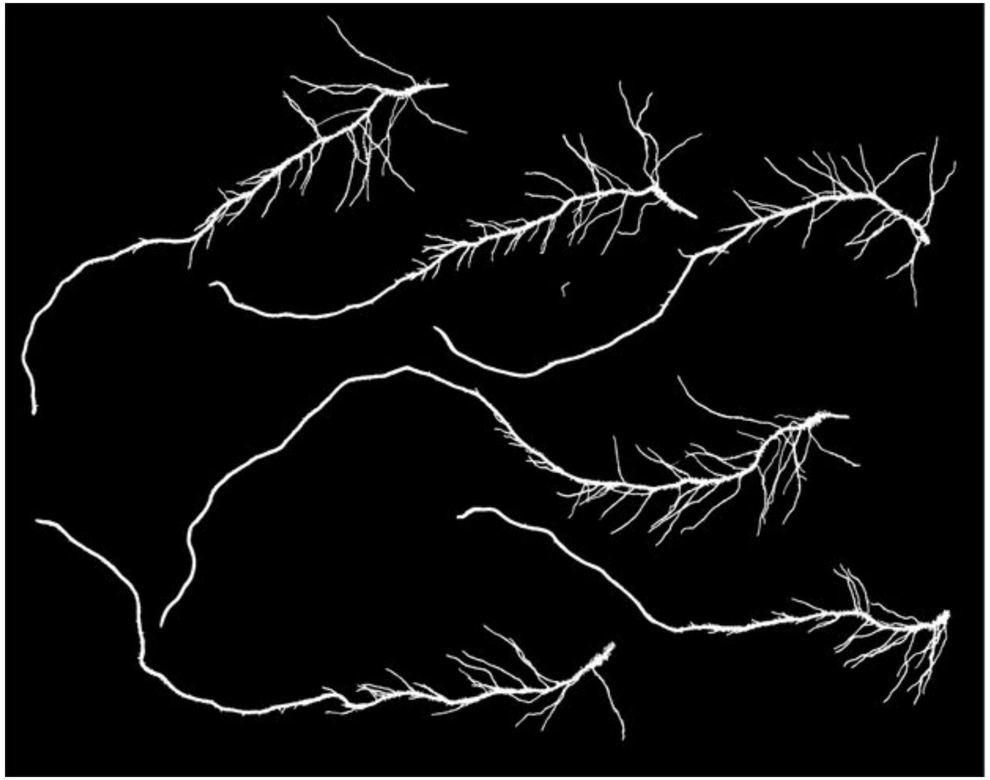


Previous approach with another possible branch correspondence



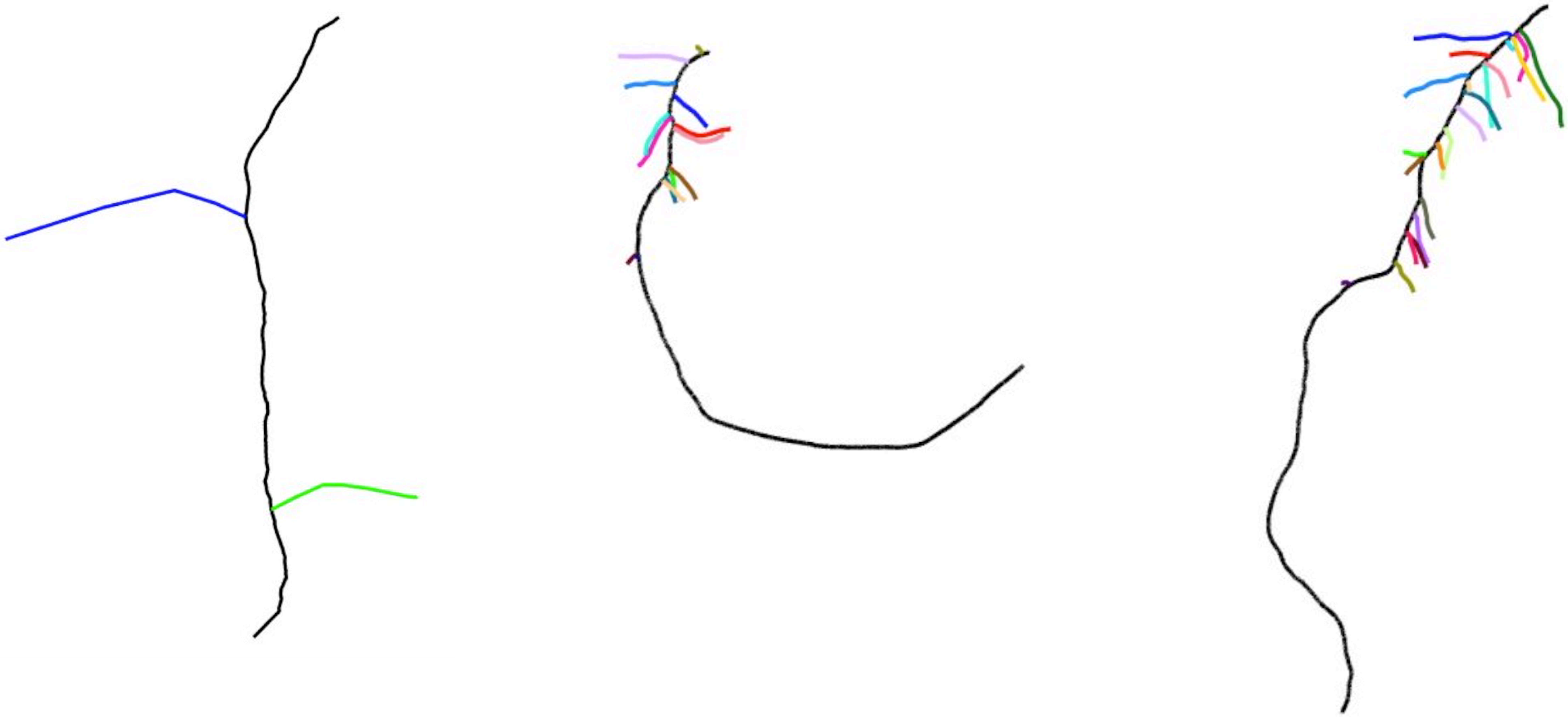
Application plant root analysis (preliminary results)

- Data set



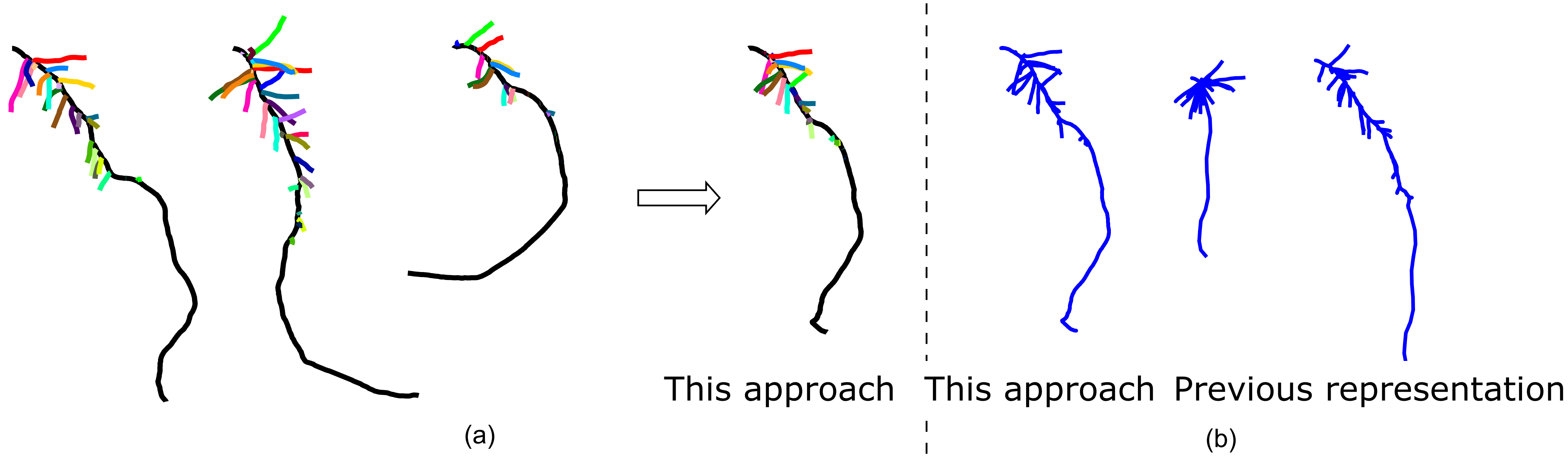
Cai et al. 2015. RootGraph: a graphic optimization tool for automated image analysis of plant roots

Examples of geodesics between wheat roots



Application to plant root analysis (preliminary results)

- Examples of mean roots



Limitations

- Computationally very expensive
  - Current implementation limited to two layers (main branch + lateral branches)
  - Extending it to more layers requires efficient implementation (note that the metric is recursive!)
- The quality of the results depends on the choice of the paremeters $\boldsymbol{\lambda} = (\lambda_m, \lambda_s, \lambda_p)$

Summary

- Two formulations of the tree-shape space and associated metric for statistical analysis of 3D botanical trees
  - The metric captures bending, stretching, and topological variations
- Used to compute
  - Geodesics, Summary statistics, random sampling
  - Regression, which is used in a 2D sketch-based 3D tree synthesis
- The second formulation is better but is computationally very expensive
  - Current implementation is limited to only two levels

Summary

- References

  - G Wang, H Laga, N Xie, J Jia, H Tabia (2018). The shape space of 3D botanical tree models. ACM Transactions on Graphics (TOG) 37 (1), 7.
  - G Wang, H Laga, J Jia, N Xie, H Tabia (2018). Statistical modeling of the 3D geometry and topology of botanical trees. Computer Graphics Forum 37 (5), 185-198

Thank You