Statistical Shape Analysis of Tree-like 3D Objects

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Statistical shape analysis

Statistical summaries
Mean and modes of variation

Represent variability with statistical models

Analyze and model growth and deformation
The building blocks

• A representation

• A metric for measuring dissimilarities (distances) between shapes
  – Is invariant to shape preserving transformations
  – Measures deformations that change shape
  – Is easy to compute

• A mechanism for computing correspondences and geodesics
  – A geodesic is an optimal sequence of deformations that align one shape onto another
Representation

Parameterization domain

$s = (u, v) \rightarrow f(s) = (x(s), y(s), z(s))$

Surface normals

\[
\begin{align*}
n(s) &= \frac{\partial f}{\partial u}(s) \times \frac{\partial f}{\partial v}(s) \\
\hat{n}(s) &= \frac{n(s)}{|n(s)|}
\end{align*}
\]

Area

\[
r(s) = |\hat{n}(s)|
\]

Curvatures

...
The difference between two surfaces is the amount of bending and stretching needed to align one surface onto the other.

Different ways of quantifying bending

- Differences in the orientation of normal vectors
- Differences in the surface curvatures
- Differences in the Second Fundamental Forms (II)

Different ways of quantifying stretch (elasticity)

- Differences in local surface area
- Differences in the First Fundamental Form (the metric)
Quantifying shape similarities and differences

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- Differences in the First Fundamental Form (the metric)

Jermyn et al. (ECCV2012) Elastic Shape Matching of Parameterized Surfaces Using Square Root Normal Fields
Only suitable for manifold shapes

- Manifold shapes are simple
  - Can be easily parameterized
  - There are elegant tools from differential geometry
  - They bend and stretch but they have a fixed topology

Laga et al. (2017) Numerical Inversion of SRNF maps for Elastic Shape Analysis. In IEEE PAMI

(a) Shape.  
(b) Pose.  
(c) Randomly synthesized 3D human bodies
What about objects that vary in topology?

- Many natural objects deform in **geometry and topology**.
In this presentation

- Objects that have a tree structure

Correspondences and geodesics

Symmetry analysis

(a) Input trees.

(b) Mean and first three modes of variation.
In this presentation

• Application to graphics (and biology)

Automatics synthesis of 3D botanical trees

Sketch-based synthesis of 3D botanical trees
In this presentation

• Building blocks
  – A representation and a metric for measuring dissimilarities
    • It should measure geometric and topological deformations that change shape
  – Correspondences and geodesics
    • Geodesic is the shortest path, with respect to a metric, between two points
    • In shape analysis, it is the optimal path that deforms one shape onto another

• Applications in graphics and biology
  – Means and modes of variation for botanical trees and plant roots
  – Synthesizing botanical trees from a few parameters (regression)
  – Symmetry analysis
The general framework
The general framework

Tree-shape space
The general framework

A path in the tree-shape space is a sequence of deformations (bending and stretching of branches, and changes in the tree topology)
The shortest path, under the metric that quantifies bending, stretching, and topological changes, is called a geodesic.
The general framework

Tree-shape space
Representation of tree-like shapes
Representation of tree-like shapes

- Tree-like shapes as a tree graph
  - Nodes $v \in V$ are bifurcation points, Edges $e \in E$ connect bifurcation points
Representation of tree-like shapes

- Each edge has a geometry
  - 1D skeletal curve + thickness of the branch at each point

Input tree \[ f_s(0), f_s(s_1), f_s(s_2), f_s(s_3) \] \( S_1, S_2, S_3 \)

Tree structure

Geometry of the edges
Representation of tree-like shapes

- Each edge has a geometry
  - 1D skeletal curve + thickness of the branch at each point

\[ x_e \in F_e = (\mathbb{R}^d)^n, \quad d = 4, \quad n \text{ is the no. of sample points} \]
Parameterization with maximal binary trees

Dotted edges are virtual (collapse) edges of length
Parameterization with maximal binary trees

A tree becomes an element of $X = \mathcal{F}_{e_1} \times \cdots \times \mathcal{F}_{e_m}$. 
The tree-shape space and the metric

• A tree is an element of

\[ X = \mathcal{F}_{e_1} \times \cdots \times \mathcal{F}_{e_m}. \]

• The tree-shape space is set of subspaces (orthants) glued together
  – A subspace \( X_k = \mathcal{F}_{e_{i_1}} \times \cdots \times \mathcal{F}_{e_{i_k}} \) contains trees whose non-zero edges are \((e_1, \cdots, e_k)\)

• Within the same orthant
  – Geometry varies but structure remains unchanged

• Transitions across orthants
  – Changes in topology (edge collapse, node split)

Topological transitions
A geodesic is the shortest path between two trees $x$ and $y$

- It can go through multiple orthants (edge collapses, node splits)
- It is the solution to

$$\arg\min \left\{ d_g^2(x, y) = \sum_{t=1}^{N-1} d^2(x_t, x_{t+1}) \right\}$$

The minimization is over all possible rigid transformations and re-parameterizations of the two trees.
The parameterization is not unique

- Binary trees that collapse to the same tree-shape are equivalent
A geodesic is the shortest path between two trees \( x \) and \( y \)
- It can go through multiple orthants (edge collapses, node splits)
- It is the solution to
\[
\text{arg min} \left\{ d^2_g(x, y) = \sum_{t=1}^{N-1} d^2(x_t, x_{t+1}) \right\}
\]

The quality of the geodesic depends on the choice of the metric \( d(\cdot, \cdot, \cdot) \)
A metric which quantifies bending and stretching

• Bending corresponds to changes in
  – The orientation of the tangent vectors to the skeletal curve

• Stretching corresponds to
  – Branch elongation, which can be measured by changes in the magnitude of the tangent vector to the skeletal curve
  – Changes in the thickness of the branches

\[ \langle \hat{\beta}, \hat{\beta} \rangle \equiv a \int \langle \theta'(s), \theta'(s) \rangle e^{\phi(s)} ds + b \int \phi'(s)^2 e^{\phi(s)} ds + c \int r'(s)^2 e^{\phi(s)} ds. \]
The ESRVF tree-shape space

- If we set
  - $a = c = 1$, $b = \frac{1}{4}$

- and define a branch with its Extended Square-Root Velocity function (ESRVF) $(q, r)$, where
  - $q$ is the square-root velocity function (SRVF) of the skeletal curve
  - $r$ is the thickness

- The complex metric becomes an L2 metric in the ESRVF space

$$ q(s) = \frac{f'(s)}{\|f'(s)\|^\frac{1}{2}}. $$
The ESRVF tree-shape space

- Map all trees onto the ESRVF space, use the QED metric to compute geodesics, and map the result back for visualization.
Examples of geodesics

QED in tree-shape space [Wang et al. ACM ToG2018]

QED in the ESRVF tree-shape space [Wang et al. SGP2018]
Examples of geodesics

QED in tree-shape space [Wang et al. ToG2018]

QED in the ESRVF tree-shape space [Wang et al. SGP2018]
Examples of geodesics (video 1 and video 2)

QEDT in tree-shape space [Wang et al. ToG2018]

QEDT in tree-shape space [This Article]

QEDT in the ESRVF tree-shape space [Wang et al. ToG2018]

QEDT in the ESRVF tree-shape space [This Article]
Application to symmetry analysis

Shape $f$
Shape $f$

Application to symmetry analysis

$\tilde{f} = H(v)f$

(Reflection of $f$ with respect to an arbitrary plane)

$H(v) = (I - 2\frac{vv^T}{v^Tv})$
Application to symmetry analysis

Shape $f$  

Fully symmetric tree

$$\tilde{f} = H(v)f$$

(Reflection of $f$ with respect to an arbitrary plane)

$$H(v) = (I - 2 \frac{vv^T}{v^Tv})$$
Application to symmetry analysis

Fully symmetric tree
Applications – Summary statistics

- Mean tree of a population of trees
  - Map all the trees to the ESRVF space
  - Compute their mean by solving
    \[ \mu_q = \arg \min_q \sum_{i=1}^{n} [d(q, q_i)]^2 \]
  - Map the result back to the tree-shape space for visualization
Applications – Summary statistics

• Modes of variation (Geodesic PCA)
  – Map all the trees to the ESRVF space
  – Compute their mean
  – Project all points to the tangent space at the mean
  – Perform PCA in the tangent space
  – Map the principal directions to principal geodesics in the ESRVF space
  – Map the principal geodesics in the ESRVF space to the tree-shape space for visualization
Applications – Random sampling

• Synthesizing random trees
  
  – Perform PCA in the tangent space to the ESRVF space
  
  – Generate random samples in this tangent space
  
  – Map the sample to the ESRVF space
  
  – Map the sample back to the trees-shape space for visualization
Application - Mean tree computation

Input tree models \( \{x_i, i = 1, \cdots, N\} \)
Application - Mean tree computation

Input tree models \( \{x_i, i = 1, \ldots, N\} \)

\[
\mu = \operatorname{arg\ min}_{x \in X} \sum_{i=1}^{N} d_{g}^2(x, x_i)
\]
Examples of mean trees

Input tree models

Mean tree
Examples of mean trees

Input tree models

Mean tree
Applications – Summary statistics

(a) Input trees.

(b) Mean and first three modes of variation.
Applications – Summary statistics

Automatically synthesized random trees

- Within one standard deviation
- Within two standard deviations
- Beyond two standard deviations
Applications – Summary statistics

(a) Input trees.

(b) Mean and first three modes of variation.
Applications – Examples of summary statistics

Automatically synthesized random trees
• Any tree can be written in the form:

\[ q = \text{Exp}_{\mu_q} \left( \sum_{i=1}^{N} b_i \cdot \Lambda_i \right) \]

• Let \( \mathbf{p} \) be a set of \( m \) parameters
  
  – Assume that any tree can be obtained from these parameters using the linear relation

\[ \mathbf{A} \times \mathbf{p} = \mathbf{b} \]

  – \( \mathbf{A} \) is the regression matrix
  – \( \mathbf{b} \) is a vector which holds the \( b_i \)’s
Applications – Regression

• Any tree can be written in the form
  \[ q = \text{Exp}_{\mu_q} \left( \sum_{i=1}^{N} b_i \cdot \Lambda_i \right) \]

• Let \( p \) be a set of \( m \) parameters
  
  – Assume that any tree can be obtained from these parameters using the linear relation
    \[ A \times \begin{bmatrix} p \end{bmatrix} = b \]

• The parameters \( b \) can be
  
  – Biologically motivated
Applications – Regression

• Any tree can be written in the form
  \[ q = \exp_{\mu_q} \left( \sum_{i=1}^{N} b_i \cdot \Lambda_i \right) \]

• Let \( p \) be a set of \( m \) parameters
  - Assume that any tree can be obtained from these parameters using the linear relation
  \[ A \times \begin{bmatrix} p \end{bmatrix} = b \]

• The parameters \( b \) can be
  - Biologically motivated
  - 2D sketches in our case
Applications – Regression (video 3 and video 4)

• Sketch-based 3D tree synthesis (videos)
Applications – Regression

- Sketch-based 3D tree synthesis
Limitation
Formulation 2

• A tree as layers of curves
  – Each side branch will grow at a location of the main parent branch
  – Each curve will be represented with its SRVF function

\[ q(s) = \frac{f'(s)}{\|f'(s)\|^\frac{1}{2}}. \]
In the case of two layers (one main branch and N lateral branches), each tree $i$ will be represented as

$$\tilde{q}^i = (\tilde{q}_0, \{ (\tilde{q}_k^i, \tilde{s}_k^i) \}_{k=1}^N)$$

- SRVF of the main branch
- Location of the $i$-th lateral branch on the main branch
- SRVF of the $i$-th lateral branch
Formulation 2

- The metric

\[
d_C(\tilde{q}^1, \tilde{q}^2) = \lambda_m \| \tilde{q}_0^1 - \tilde{q}_0^2 \|^2 + \lambda_s \sum_{k=1}^{N} \| \tilde{q}_k^1 - \tilde{q}_k^2 \|^2 + \lambda_p \sum_{k=1}^{N} (\tilde{s}_k^1 - \tilde{s}_k^2)^2
\]
Formulation 2

• The metric

• The parameters $\lambda = (\lambda_m, \lambda_s, \lambda_p)$ control the cost of
  
  – Deforming the main branch
  
  – Deforming the lateral branches
  
  – Moving the position of the lateral branches along their parent branch

$$d_C(\tilde{q}^1, \tilde{q}^2) = \lambda_m \| \tilde{q}_0^1 - \tilde{q}_0^2 \|^2 + \lambda_s \sum_{k=1}^{N} \| \tilde{q}_k^1 - \tilde{q}_k^2 \|^2 + \lambda_p \sum_{k=1}^{N} (\tilde{s}_k^1 - \tilde{s}_k^2)^2$$
Some preliminary results

This approach

Previous approach with one possible branch correspondence

Previous approach with another possible branch correspondence
Application plant root analysis (preliminary results)

- Data set

Cai et al. 2015. RootGraph: a graphic optimization tool for automated image analysis of plant roots
Examples of geodesics between wheat roots
Application to plant root analysis (preliminary results)

- Examples of mean roots

![Diagram](image)

- This approach
- Previous representation
Limitations

• Computationally very expensive
  – Current implementation limited to two layers (main branch + lateral branches)
  – Extending it to more layers requires efficient implementation (note that the metric is recursive!)

• The quality of the results depends on the choice of the parameters \( \mathbf{\lambda} = (\lambda_m, \lambda_s, \lambda_p) \)
Two formulations of the tree-shape space and associated metric for statistical analysis of 3D botanical trees
- The metric captures bending, stretching, and topological variations

Used to compute
- Geodesics, Summary statistics, random sampling
- Regression, which is used in a 2D sketch-based 3D tree synthesis

The second formulation is better but is computationally very expensive
- Current implementation is limited to only two levels
• References


Thank You

Input trees

Mean tree