An Algebraic Proof of Cut Elimination

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January 25, 2010
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Why an algebraic proof of cut elimination?

- To clarify the meaning of cut elimination from an algebraic point of view.
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- To provide a proof of cut elimination comprehensible to algebraists, which avoids heavy syntactic arguments.
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- These slides are adapted from the talk given by prof. Ono at the Logic Summer School, ANU, December 2004.
We introduce Gentzen structures for the sequent system $\text{FL}_{\text{ew}}$ without cut. $\text{FL}_{\text{ew}}$ is intuitionistic logic without the contraction rule.
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- We use the *quasi-completion* of these Gentzen structures to show the completeness of $\text{FL}_{\text{ew}}$ without cut with respect to $\text{FL}_{\text{ew}}$-algebras.
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We use the quasi-completion of these Gentzen structures to show the completeness of $\textbf{FL}_{ew}$ without cut with respect to $\textbf{FL}_{ew}$-algebras.

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In the process we show that the quasi-completion is a generalization of the MacNeille completion.
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- We introduce *Gentzen structures* for the sequent system $\text{FL}_{\text{ew}}$ without cut. $\text{FL}_{\text{ew}}$ is intuitionistic logic without the contraction rule.
- We use the *quasi-completion* of these Gentzen structures to show the completeness of $\text{FL}_{\text{ew}}$ without cut with respect to $\text{FL}_{\text{ew}}$-algebras.
- This method works for a variety of sequent systems of nonclassical (substructural, modal) logic, both in the propositional and predicate case.
- In the process we show that the quasi-completion is a generalization of the MacNeille completion.
- Moreover, the finite model property is obtained for many cases by modifying our completeness proof.
The sequent calculus $\text{FL}_{\text{ew}}$

The sequent calculus $\text{FL}_{\text{ew}}$ is obtained from intuitionistic logic $\text{LJ}$ by deleting the contraction rule.

**Initial sequents:** 1) $\alpha \Rightarrow \alpha$, 2) $0 \Rightarrow$, 3) $\Rightarrow 1$.

**Logical rules:**

- $\frac{\Gamma \Rightarrow \delta}{1, \Gamma \Rightarrow \delta}$ (1 ⇒)
- $\frac{\Gamma \Rightarrow 0}{\Gamma \Rightarrow 0}$ (⇒ 0)
- $\frac{\Gamma \Rightarrow \alpha \quad \beta, \Sigma \Rightarrow \delta}{\alpha \rightarrow \beta, \Gamma, \Sigma \Rightarrow \delta}$ (→⇒)
- $\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}$ (⇒→)
- $\frac{\alpha, \Gamma \Rightarrow \delta}{\alpha \land \beta, \Gamma \Rightarrow \delta}$ (∧1 ⇒)
- $\frac{\beta, \Gamma \Rightarrow \delta}{\alpha \land \beta, \Gamma \Rightarrow \delta}$ (∧2 ⇒)
- $\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \land \beta}$ (⇒ ∧)
- $\frac{\alpha, \Gamma \Rightarrow \delta \quad \beta, \Gamma \Rightarrow \delta}{\alpha \lor \beta, \Gamma \Rightarrow \delta}$ (∨ ⇒)
- $\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \lor \beta}$ (⇒ ∨1)
- $\frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \lor \beta}$ (⇒ ∨2)
- $\frac{\alpha, \beta, \Gamma \Rightarrow \delta}{\alpha \cdot \beta, \Gamma \Rightarrow \delta}$ (⇒)
- $\frac{\Gamma \Rightarrow \alpha \quad \Sigma \Rightarrow \beta}{\Gamma, \Sigma \Rightarrow \alpha \cdot \beta}$ (⇒ ·)
The sequent calculus $\text{FL}_{ew}$

Structural rules:

$$
\begin{align*}
& \frac{\Gamma \Rightarrow \delta}{\alpha, \Gamma \Rightarrow \delta} \quad (w \Rightarrow) & & \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha} \quad (\Rightarrow w) \\
& \frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \delta}{\Gamma, \beta, \alpha, \Sigma \Rightarrow \delta} \quad (e \Rightarrow) & & \frac{\Gamma \Rightarrow \alpha, \Sigma \Rightarrow \delta}{\Gamma, \Sigma \Rightarrow \delta} \quad (cut)
\end{align*}
$$
The sequent calculus $\text{FL}_{ew}$

Structural rules:

\[
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\frac{\Gamma \Rightarrow \delta}{\alpha, \Gamma \Rightarrow \delta} \quad (w \Rightarrow) & \quad \frac{\Gamma \Rightarrow \delta}{\Gamma \Rightarrow \alpha} \quad (\Rightarrow w) \\
\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \delta}{\Gamma, \beta, \alpha, \Sigma \Rightarrow \delta} \quad (e \Rightarrow) & \quad \frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \delta}{\Gamma, \Sigma \Rightarrow \delta} \quad (\text{cut})
\end{align*}
\]

Theorem (Cut elimination [4])

If a sequent $\Gamma \Rightarrow \delta$ is provable in $\text{FL}_{ew}$ then it is provable in $\text{FL}_{ew}^-$ without using the cut rule.

$\text{FL}_{ew}^-$ denotes the sequent system obtained from $\text{FL}_{ew}$ by deleting the cut rule.
Definition
A structure $P = \langle P, \land, \lor, \cdot, \rightarrow, 0, 1 \rangle$ is a $\text{FL}_{\text{ew}}$-algebra if:

1. $\langle P, \land, \lor, 0, 1 \rangle$ is a bounded lattice,
2. $\langle P, \cdot, 1 \rangle$ is a commutative monoid with the unit 1,
3. $a \cdot b \leq c$ iff $a \leq (b \rightarrow c)$ (law of residuation).
**FL\textsubscript{ew}-algebras**

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Let $h$ be an assignment of propositional variables to elements of $P$ such that $h(0) = 0$ and $h(1) = 1$.
The assignment $h$ can be lifted to the set of all formulas.
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**Definition**
A sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ is valid on an $\text{FL}\textsubscript{ew}$-algebra $P$ iff $h(\alpha_1) \cdot \ldots \cdot h(\alpha_n) \leq h(\beta)$ holds in $P$ for any assignment $h$. 
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**Definition**
A sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ is valid on an FL\textsubscript{ew}-algebra $\mathbf{P}$ iff $h(\alpha_1) \cdot \ldots \cdot h(\alpha_n) \leq h(\beta)$ holds in $\mathbf{P}$ for any assignment $h$.

**Theorem (Completeness of FL\textsubscript{ew})**
A sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable in FL\textsubscript{ew} iff it is valid on every FL\textsubscript{ew}-algebra.
Gentzen structures for $\text{FL}_{\text{ew}}$

For a nonempty set $Q$, let $Q^*$ be the set of all (finite, possibly empty) multisets of members of $Q$. The empty multiset is denoted by $\varepsilon$. 
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For a nonempty set $Q$, let $Q^*$ be the set of all (finite, possibly empty) multisets of members of $Q$. The empty multiset is denoted by $\varepsilon$.

A *Gentzen structure* for $\text{FL}_{\text{ew}}$ is a tuple $Q = \langle Q, \preceq, \land, \lor, \cdot, \to, 0, 1 \rangle$ such that $0, 1 \in Q$, $\land, \lor, \cdot, \to$ are binary operations on $Q$, and $\preceq$ is a subset of $Q^* \times (Q \cup \{\varepsilon\})$ that satisfies the following conditions:

- $a \preceq a$ and $0 \preceq c$ and $\varepsilon \preceq 1$
- $x \preceq c$ implies $dx \preceq c$
- $x \preceq a$ and $by \preceq c$ imply $(a \to b)xy \preceq c$
- $ax \preceq b$ implies $x \preceq a \to b$
- $ax \preceq c$ and $bx \preceq c$ imply $(a \lor b)x \preceq c$
- $x \preceq a$ implies $x \preceq a \lor b$
- $x \preceq b$ implies $x \preceq a \lor b$
- $ax \preceq c$ implies $(a \land b)x \preceq c$
- $bx \preceq c$ implies $(a \land b)x \preceq c$
- $x \preceq a$ and $x \preceq b$ imply $x \preceq a \land b$
- $abx \preceq c$ implies $(a \cdot b)x \preceq c$
- $x \preceq a$ and $y \preceq b$ imply $xy \preceq a \cdot b$
Let $g$ be an assignment of propositional variables to elements in $Q$ such that $g(0) = 0$ and $g(1) = 1$. The assignment $g$ can be lifted to the set of all formulas.
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**Definition**

A sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ is valid on a Gentzen structure $Q$ iff $\langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \preceq g(\beta)$ holds in $Q$ for any assignment $g$. 
Gentzen structures for $\text{FL}_{ew}$

Let $g$ be an assignment of propositional variables to elements in $Q$ such that $g(0) = 0$ and $g(1) = 1$.

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**Definition**

A sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ is valid on a Gentzen structure $Q$ iff

$\langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \preceq g(\beta)$ holds in $Q$ for any assignment $g$.

The system $\text{FL}_{ew}^-$ is complete with respect to the class of Gentzen structures.

**Theorem**

A sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable in $\text{FL}_{ew}^-$ iff it is valid on every Gentzen structure.
Each $\text{FL}_{ew}$-algebra can be seen as a Gentzen structure if $\preceq$ is defined by

$$\langle a_1, \ldots, a_m \rangle \preceq c \quad \text{iff} \quad (a_1 \cdot \ldots \cdot a_m) \leq c$$
Each $\text{FL}_{ew}$-algebra can be seen as a Gentzen structure if $\preceq$ is defined by
\[
\langle a_1, \ldots, a_m \rangle \preceq c \quad \text{iff} \quad (a_1 \cdot \ldots \cdot a_m) \leq c
\]

Also, let $Q$ be any Gentzen structure with a strongly transitive $\preceq$:

$x \preceq a$ and $ay \preceq c$ imply $xy \preceq c$

If the restriction $\preceq_0$ of $\preceq$ to $Q \times Q$ is moreover antisymmetric, then $Q$ is a $\text{FL}_{ew}$-algebra with the lattice order $\preceq_0$. 
Gentzen structures and $\text{FL}_{ew}$-algebras

- Each $\text{FL}_{ew}$-algebra can be seen as a Gentzen structure if $\leq$ is defined by

  $$\langle a_1, \ldots, a_m \rangle \leq c \iff (a_1 \cdot \ldots \cdot a_m) \leq c$$

- Also, let $Q$ be any Gentzen structure with a strongly transitive $\leq$:

  $$x \leq a \text{ and } ay \leq c \text{ imply } xy \leq c$$

  If the restriction $\leq_0$ of $\leq$ to $Q \times Q$ is moreover antisymmetric, then $Q$ is a $\text{FL}_{ew}$-algebra with the lattice order $\leq_0$.

- In conclusion, we can say that any Gentzen structure with a strongly transitive relation can be identified with a $\text{FL}_{ew}$-algebra, and vice versa.
To prove cut elimination for $\mathbf{FL}_{ew}$ it is enough to show the following result:

**Lemma**

If $\langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \preceq g(\beta)$ fails for some $g$ in a Gentzen structure $Q$ then $h(\alpha_1) \cdot \ldots \cdot h(\alpha_n) \leq h(\beta)$ fails for some $h$ in an $\mathbf{FL}_{ew}$-algebra $P$. 
To prove cut elimination for $\text{FL}_{\text{ew}}$ it is enough to show the following result:

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How do we get such an $\text{FL}_{\text{ew}}$-algebra $P$ from a given Gentzen structure $Q$? Moreover, $Q$ must be embedded into the $\text{FL}_{\text{ew}}$-algebra $P$.

1. We give a uniform way of constructing such a $P$ called the *quasi-completion* of $Q$;
2. We show that $Q$ can be quasi-embedded into $P$. 

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*An Algebraic Proof of Cut Elimination*
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How do we get such an $\text{FL}_{ew}$-algebra $P$ from a given Gentzen structure $Q$? Moreover, $Q$ must be embedded into the $\text{FL}_{ew}$-algebra $P$.

1. We give a uniform way of constructing such a $P$ called the *quasi-completion* of $Q$;
2. We show that $Q$ can be quasi-embedded into $P$.

When $Q$ is a $\text{FL}_{ew}$-algebra, $P$ is a MacNeille completion of $Q$ and the quasi-embedding becomes a complete embedding.
Let $\mathbf{M} = \langle M, \cdot, 1 \rangle$ be a commutative monoid. A unary function $C$ on $\wp(M)$ is a closure operator if for all $X, Y \in \wp(M)$:

1. $X \subseteq C(X)$
2. $C(C(X)) \subseteq C(X)$
3. $X \subseteq Y$ implies $C(X) \subseteq C(Y)$
4. $C(X) \ast C(Y) \subseteq C(X \ast Y)$, where $W \ast Z = \{w \cdot z \mid w \in W \text{ and } z \in Z\}$. 

Lemma

The tuple $\mathbf{C} \mathbf{M} = \langle \mathcal{C}(\wp(M)), \cap, \cup, \ast C, \Rightarrow, \mathcal{C}(\emptyset), \mathcal{C}(\{1\}) \rangle$ is a FL$_{e}$-algebra, not necessarily integral.
Closure operators

Let $\mathbf{M} = \langle M, \cdot, 1 \rangle$ be a commutative monoid.
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Let $C(\wp(M))$ be the set of all $C$-closed subsets, define operations $\cup_C, \ast_C$ and $\Rightarrow$ on $C(\wp(M))$ as follows:

- $X \cup_C Y = C(X \cup Y)$
- $X \ast_C Y = C(X \ast Y)$
- $X \Rightarrow Y = \{z \mid X \ast \{z\} \subseteq Y\}$
Closure operators

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Lemma

The tuple $\mathbf{C}_M = \langle C(\wp(M)), \cap, \cup_C, \ast_C, \Rightarrow, C(\emptyset), C(\{1\}) \rangle$ is a FL$_e$-algebra, not necessarily integral.
Quasi-completions

Let $Q$ be a Gentzen structure, for $x \in Q^*$ and $a \in Q \cup \{\varepsilon\}$ define

$$[x; a] = \{ w \in Q^* \mid xw \preceq a \}$$
Quasi-completions

Let $Q$ be a Gentzen structure, for $x \in Q^*$ and $a \in Q \cup \{\varepsilon\}$ define

$$[x; a] = \{ w \in Q^* \mid xw \leq a \}$$

Now define a function $C$ on $\wp(Q^*)$ by

$$C(X) = \bigcap \{ [x; a] \mid X \subseteq [x; a] \text{ for } x \in Q^* \text{ and } a \in Q \cup \{\varepsilon\} \}$$
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The function $C$ is a closure operator such that $C(\{\varepsilon\}) = Q^* = C(\{1\})$. Thus, $C_{Q^*}$ is a $\mathbf{FL}_{ew}$-algebra, which is called the quasi-completion of $Q$. 
To show that the Gentzen structure $Q$ is quasi-embeddable into $C_{Q^*}$ we define a *quasi-embedding* $k : Q \rightarrow C(\varnothing(Q^*))$ as

$$k(a) = [\varepsilon; a] = \{ w \in Q^* | w \sqsubseteq a \}$$
Quasi-embeddings

- To show that the Gentzen structure $Q$ is quasi-embeddable into $C_{Q^*}$ we define a *quasi-embedding* $k : Q \rightarrow C(\wp(Q^*))$ as

$$k(a) = [\varepsilon; a] = \{ w \in Q^* \mid w \preceq a \}$$

- Then we can prove the following.

**Lemma**

Suppose that $a, b \in Q$ and that $U$ and $V$ are arbitrary $C$-closed subsets of $Q^*$ such that $a \in U \subseteq k(a)$ and $b \in V \subseteq k(b)$, then for each $\star \in \{\land, \lor, \cdot, \rightarrow\}$:

$$a \star b \in U \star_C V \subseteq k(a \star b),$$

where $\star_C$ denotes $\cap, \cup_C, \ast_C$ and $\Rightarrow$ respectively.
Suppose that \( \langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \preceq g(\beta) \) does not hold in \( Q \) by an assignment \( g \).
Suppose that $\langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \preceq g(\beta)$ does not hold in $Q$ by an assignment $g$.

Define an assignment $h$ on $C_{Q^*}$ as $h(q) = k(g(q))$ for each proposition $q$. 
Suppose that $\langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \preceq g(\beta)$ does not hold in $Q$ by an assignment $g$.

Define an assignment $h$ on $C_{Q^*}$ as $h(q) = k(g(q))$ for each proposition $q$.

By induction on the length of a formula $\phi$ we can show that:

$$g(\phi) \in h(\phi) \subseteq k(g(\phi))$$
Proof of cut elimination - concluded

- Suppose that $\langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \subseteq g(\beta)$ does not hold in $Q$ by an assignment $g$.
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- Now, suppose that $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ holds in $C_Q^*$. 

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An Algebraic Proof of Cut Elimination
Proof of cut elimination - concluded

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- Now, suppose that $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ holds in $C_{Q^*}$.
- Then in particular $h(\alpha_1) \ast_C \ldots \ast_C h(\alpha_n) \subseteq h(\beta)$, and by the results above,

$$\langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \in h(\alpha_1) \ast_C \ldots \ast_C h(\alpha_n) \subseteq h(\beta) \subseteq k(g(\beta)) = \{w \mid w \preceq g(\beta)\}$$
Proof of cut elimination - concluded

▶ Suppose that $\langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \preceq g(\beta)$ does not hold in $Q$ by an assignment $g$.
▶ Define an assignment $h$ on $CQ^*$ as $h(q) = k(g(q))$ for each proposition $q$.
▶ By induction on the length of a formula $\phi$ we can show that:

$$g(\phi) \in h(\phi) \subseteq k(g(\phi))$$

▶ Now, suppose that $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ holds in $CQ^*$.
▶ Then in particular $h(\alpha_1) \ast_C \ldots \ast_C h(\alpha_n) \subseteq h(\beta)$, and by the results above,

$$\langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \in h(\alpha_1) \ast_C \ldots \ast_C h(\alpha_n) \subseteq h(\beta) \subseteq k(g(\beta)) = \{ w \mid w \preceq g(\beta) \}$$

▶ But this implies $\langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \preceq g(\beta)$, which is a contradiction. Thus, $\alpha_1, \ldots \alpha_n \Rightarrow \beta$ is not valid in $CQ^*$. 

Proof of cut elimination - concluded

- Suppose that \( \langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \preceq g(\beta) \) does not hold in \( Q \) by an assignment \( g \).

- Define an assignment \( h \) on \( C_Q^* \) as \( h(q) = k(g(q)) \) for each proposition \( q \).

- By induction on the length of a formula \( \phi \) we can show that:

  \[
  g(\phi) \in h(\phi) \subseteq k(g(\phi))
  \]

- Now, suppose that \( \alpha_1, \ldots, \alpha_n \Rightarrow \beta \) holds in \( C_Q^* \).

- Then in particular \( h(\alpha_1) \ast_C \ldots \ast_C h(\alpha_n) \subseteq h(\beta) \), and by the results above,

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  \langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \in h(\alpha_1) \ast_C \ldots \ast_C h(\alpha_n) \subseteq h(\beta) \subseteq k(g(\beta)) = \{ w \mid w \preceq g(\beta) \}
  \]

- But this implies \( \langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \preceq g(\beta) \), which is a contradiction. Thus, \( \alpha_1, \ldots, \alpha_n \Rightarrow \beta \) is not valid in \( C_Q^* \).

- This completes the proof of cut elimination for \( FL_{ew} \).
Let $P$ be a $\text{FL}_{ew}$-algebra and define

$$C(X) = \bigcap \{[x; a] \mid X \subseteq [x; a] \text{ for } x \in P^* \text{ and } a \in P \cup \{e\}\}$$
MacNeille- and Quasi-completions

- Let $P$ be a $\text{FL}_{\text{ew}}$-algebra and define
  \[ C(X) = \bigcap \{ [x; a] \mid X \subseteq [x; a] \text{ for } x \in P^* \text{ and } a \in P \cup \{ \varepsilon \} \} \]
- Then we can show that
  \[ C(X) = (X \rightarrow) \leftarrow = \{ a \mid a \leq b \text{ for all } b \text{ such that } b \geq c \text{ for all } c \in X \} \]
  and therefore the quasi-completion $C_{P^*}$ of $P$ is isomorphic to the MacNeille completion of $P$. 
MacNeille- and Quasi-completions

Let $P$ be a $\text{FL}_{\text{ew}}$-algebra and define

$$C(X) = \bigcap \{ [x; a] \mid X \subseteq [x; a] \text{ for } x \in P^* \text{ and } a \in P \cup \{ \varepsilon \} \}$$

Then we can show that

$$C(X) = (X \rightarrow) \leftarrow = \{ a \mid a \leq b \text{ for all } b \text{ such that } b \geq c \text{ for all } c \in X \}$$

and therefore the quasi-completion $C_{P^*}$ of $P$ is isomorphic to the MacNeille completion of $P$.

Further, since $\leq$ is strongly transitive, $a \star b \in k(a) \star_C k(b)$ implies that $k(a \star b) = k(a) \star_C k(b)$. 

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Thus, the map $k$ can be identified with the complete embedding of a $\mathbf{FL}_{ew}$-algebra $P$ into its MacNeille completion.
Extensions to other systems

This algebraic proof of cut elimination can be extended to:

- the intuitionistic and classic systems \( \text{LJ} \) and \( \text{LK} \).
- intuitionistic substructural systems:
  - propositional calculi \( \text{FL}_e \) and \( \text{FL}_{ec} \).
  - first-order calculi \( \text{QFL}_{ew}, \text{QFL}_e \) and \( \text{QFL}_{ec} \).
- the classic substructural systems \( \text{CFL}_{ew}, \text{CFL}_e \) and \( \text{CFL}_{ec} \).
- the propositional modal logics \( \text{K}, \text{T} \) and \( \text{S4} \).
The Finite Model Property

The idea of the proof is based on [2, 3].

**Lemma**

Let $Q$ be a Gentzen structure for $\text{FL}_{ew}$ such that the closed base $B = \{ [x; a] \mid x \in Q^*, a \in Q \cup \{ \varepsilon \} \}$ is finite, then the quasi-completion $C_{Q^*}$ of $Q$ is also finite.
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Now define a subset $P_{(x,a)}$ of $Q^* \times (Q \cup \{\varepsilon\})$ such that:

1. $(x, a) \in P_{(x,a)}$.
2. Suppose that $(w, b) \in P_{(x,a)}$. If "$u \preceq c$ implies $w \preceq b$" is one of the conditions for $\preceq$ in $Q$, then $(u, c)$ is a member of $P_{(x,a)}$. Similarly, if "$u \preceq c$ and $v \preceq d$ imply $w \preceq b$" is one of the conditions for $\preceq$. 


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For a finite subset $S$ of $Q^* \times (Q \cup \{\varepsilon\})$, let $P_S$ be the union of $P_{(x, a)}$ for $(x, a) \in S$. We say that the set $S$ is *finitely based*, when $P_S$ is finite.
The Finite Model Property

Now we show how to obtain a Gentzen structure for $\text{FL}_{ew}$ such that the closed base $B$ is finite.

**Lemma**

If $Q$ is a Gentzen structure for $\text{FL}_{ew}$ and $S$ is finitely based, then the relation $\preceq^*$ such that for $(w, b) \in \mathcal{P}_S$, $w \preceq^* b$ iff $w \preceq b$, and otherwise $w \preceq^* b$ always holds, satisfies the following conditions:

1. the structure $Q^* = \langle Q, \preceq^*, \land, \lor, \cdot, \rightarrow, 0, 1 \rangle$ is a Gentzen structure for $\text{FL}_{ew}$.
2. the closed base $B$ determined by $\preceq^*$ is finite.
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If $Q$ is a Gentzen structure for $\text{FL}_{ew}$ and $S$ is finitely based, then the relation $\preceq^*$ such that for $(w, b) \in \mathcal{P}_S$, $w \preceq^* b$ iff $w \preceq b$, and otherwise $w \preceq^* b$ always holds, satisfies the following conditions:

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The finite model property for $\mathbf{FL}_{\text{ew}}$

- Suppose that $\mathbf{FL}_{\text{ew}} \not\vdash \alpha_1, \ldots, \alpha_m \Rightarrow \beta$, then $\langle \alpha_1, \ldots, \alpha_m \rangle \preceq \beta$ doesn’t hold in the *free* Gentzen structure $Q^+$ for $\mathbf{FL}_{\text{ew}}$. 

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- Suppose that $\text{FL}_{\text{ew}} \not\vdash \alpha_1, \ldots, \alpha_m \Rightarrow \beta$, then $\langle \alpha_1, \ldots, \alpha_m \rangle \leq \beta$ doesn't hold in the free Gentzen structure $Q^+$ for $\text{FL}_{\text{ew}}$.
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- By the lemma above, $\{(\langle \alpha_{1}, \ldots, \alpha_{m} \rangle, \beta)\}$ is embedded into a Gentzen structure $(Q^{+})^{*}$ for $\text{FL}_{\text{ew}}$ with a relation $\preceq^{*}$ such that the closed base is finite. Moreover, $\langle \alpha_{1}, \ldots, \alpha_{m} \rangle \preceq^{*} \beta$ doesn’t hold in $(Q^{+})^{*}$ by definition.
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- By a previous lemma, the quasi-completion $R$ of $(Q^+)^*$ is finite. Since $\langle \alpha_1, \ldots, \alpha_m \rangle \leq^* \beta$ doesn’t hold in $(Q^+)^*$, $(\alpha_1 \cdot \ldots \cdot \alpha_m) \leq \beta$ doesn’t hold either in $R$, which is a $\text{FL}_{ew}$-algebra.
Suppose that $\text{FL}_{\text{ew}} \not \vdash \alpha_1, \ldots, \alpha_m \Rightarrow \beta$, then $\langle \alpha_1, \ldots, \alpha_m \rangle \preceq \beta$ doesn’t hold in the free Gentzen structure $Q^+$ for $\text{FL}_{\text{ew}}$.

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By the lemma above, $\{(\langle \alpha_1, \ldots, \alpha_m \rangle, \beta)\}$ is embedded into a Gentzen structure $(Q^+)^*$ for $\text{FL}_{\text{ew}}$ with a relation $\preceq^*$ such that the closed base is finite. Moreover, $\langle \alpha_1, \ldots, \alpha_m \rangle \preceq^* \beta$ doesn’t hold in $(Q^+)^*$ by definition.

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This proof of the finite model property can be extended to the first-order substructural logic $\text{QFL}_{\text{ew}}$. 

Thank you!

