Counterpart Semantics at work: An Incompleteness Result in Quantified Modal Logic
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ABSTRACT. In this paper we make use of counterpart semantics to prove an original incompleteness result in quantified modal logic (QML), that is, the system $Q^E.K+BF$ based on free logic and containing the Barcan formula is incomplete with respect to Kripke semantics. This incompleteness result extends to the system $Q^E.K+CBF+BF$ obtained by adding the converse of the Barcan formula to $Q^E.K+BF$.

Keywords: Quantified Modal Logic, Kripke and Counterpart Semantics, Incompleteness.

1 Kripke Semantics
In this paper we consider a first-order modal alphabet $A$ containing a denumerable infinite set $\text{Var}$ of individual variables $x_1, x_2, \ldots$; a denumerable infinite set of $n$-ary predicative constants $P^1_n, P^2_n, \ldots$, for $n \in \mathbb{N}$; the connectives $\neg$ and $\rightarrow$; the quantifier $\forall$; the operator $\Box$; the existence predicative constant $E$. The terms $t_1, t_2, \ldots$ are only individual variables.

DEFINITION 1. The formulas in the first-order modal language $L$ are defined in the Backus-Naur form as follows:

$$\phi ::= P^n(t_1, \ldots, t_n) | E(t) | \neg \phi | \phi \rightarrow \psi | \forall x \phi | \Box \phi$$

The symbols $\land$, $\lor$, $\leftrightarrow$, $\exists$, $\diamond$ are standardly defined; $\phi[\vec{y}/\vec{t}]$ denotes the simultaneous substitution of some, possibly all, free occurrences of $\vec{y} = y_1, \ldots, y_n$ in $\phi$ with $\vec{t} = t_1, \ldots, t_n$, renaming bounded variables if necessary.

DEFINITION 2. A Kripke frame, or $K$-frame, is a tuple $F = \langle W, R, D, d \rangle$ such that $W$ is a non-empty set; $R \subseteq W^2$; for $w, w' \in W$, $D(w)$ is a non-empty set and $wRw'$ implies $D(w) \subseteq D(w')$; for $w \in W$, $d(w) \subseteq D(w)$.

A $K$-frame $F$ has constant (resp. increasing, decreasing) inner domains iff $wRw'$ implies $d(w) = d(w')$ (resp. $d(w) \subseteq d(w')$, $d(w) \supseteq d(w')$).

DEFINITION 3. A Kripke model of language $L$ based on a $K$-frame $F$, or $K$-model, is a pair $M = \langle F, I \rangle$ where $I$ is an interpretation of $L$ such that (i) if $P^n$ is an $n$-ary predicative constant and $w \in W$, then $I(P^n, w)$ is an $n$-ary relation on $D(w)$; (ii) $I(E, w) = d(w)$.

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A \( w \)-assignments is any function \( \sigma : \text{Var} \to D(w) \). The variant \( \sigma(x) \) does not coincide with \( \sigma \) at most on \( x \), and assigns \( a \in D(w) \) to \( x \).

**DEFINITION 4.** The satisfaction relation \( \models \) for a world \( w \in \mathcal{M} \), a formula \( \phi \in \mathcal{L} \), and a \( w \)-assignment \( \sigma \) is defined as follows:

\[
\begin{align*}
(\mathcal{M}^\sigma, w) & \models P^\sigma(x_1, \ldots, x_n) \text{ iff } (\sigma(t_1), \ldots, \sigma(t_n)) \in I(P^\sigma, w) \\
(\mathcal{M}^\sigma, w) & \models \neg \psi \text{ iff } (\mathcal{M}^\sigma, w) \not\models \psi \\
(\mathcal{M}^\sigma, w) & \models \psi \rightarrow \psi' \text{ iff } (\mathcal{M}^\sigma, w) \not\models \psi \text{ or } (\mathcal{M}^\sigma, w) \models \psi' \\
(\mathcal{M}^\sigma, w) & \models \forall x \phi \text{ iff for every } w' \in W, wRw' \text{ implies } (\mathcal{M}^\sigma, w') \models \phi \\
(\mathcal{M}^\sigma, w) & \models \exists x \phi \text{ iff for every } a \in D(w), (\mathcal{M}^\sigma(a), w) \models \phi
\end{align*}
\]

A formula \( \phi \) is true at a world \( w \) iff it is satisfied by every \( w \)-assignment \( \sigma \); \( \phi \) is valid on a \( K \)-model \( \mathcal{M} \) iff it is true at every world in \( \mathcal{M} \); \( \phi \) is valid on a \( K \)-frame \( \mathcal{F} \) iff it is valid on every \( K \)-model based on \( \mathcal{F} \).

**2 The Systems \( Q^E.K+BF \) and \( Q^E.K+CBF+BF \)**

We now introduce the systems \( Q^E.K+BF \) and \( Q^E.K+CBF+BF \) based on free logic. We will consider the following principles in what follows.

<table>
<thead>
<tr>
<th>Calculi</th>
<th>Inner Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Taut )</td>
<td>Tautologies of classical propositional calculus</td>
</tr>
<tr>
<td>( K )</td>
<td>( \Box(\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi) )</td>
</tr>
<tr>
<td>( MP )</td>
<td>( \phi \rightarrow \psi, \phi \vdash \psi )</td>
</tr>
<tr>
<td>( Nec )</td>
<td>( \phi \vdash \Box \phi )</td>
</tr>
<tr>
<td>( E-Ex )</td>
<td>( \forall x \phi \rightarrow (E(y) \rightarrow \phi(x/y)) )</td>
</tr>
<tr>
<td>( E-Gen )</td>
<td>( \phi \rightarrow (E(x) \rightarrow \psi) \Rightarrow \phi \rightarrow \forall x \psi, x \text{ not free in } \phi )</td>
</tr>
<tr>
<td>( BF )</td>
<td>( \forall x \Box \phi \rightarrow \Box \forall x \phi )</td>
</tr>
<tr>
<td>( CBF )</td>
<td>( \Box \forall x \phi \rightarrow \forall x \Box \phi )</td>
</tr>
<tr>
<td>( N-\Box )</td>
<td>( \neg E(x) \rightarrow \Box \neg E(x) )</td>
</tr>
<tr>
<td>( NE )</td>
<td>( E(x) \rightarrow \Box E(x) )</td>
</tr>
</tbody>
</table>

**DEFINITION 5.** The system \( Q^E.K+BF \) includes the schemes of axioms \( Taut, K, E-Ex, BF \), and the inference rules \( MP, Nec, E-Gen \). The system \( Q^E.K+CBF+BF \) extends \( Q^E.K+BF \) by adding \( CBF \).

We consider the standard definitions of **proof** and **theorem**: \( S \vdash \phi \) means that \( \phi \) is a theorem in the system \( S \). A \( K \)-frame \( \mathcal{F} \) is a \( K \)-frame for \( \phi \) iff all the theorems of \( S \) are valid on \( \mathcal{F} \), i.e., \( S \vdash \phi \) implies \( \mathcal{F} \models \phi \).

**LEMMA 6.** For any system \( S \) in the first column, \( \mathcal{F} \) is a \( K \)-frame for \( \phi \) iff it satisfies the constraint on inner domains in the second column:

\[
\begin{array}{|c|c|}
\hline
\text{Calculi} & \text{Inner Domain} \\
\hline
Q^E.K+BF & \text{decreasing} \\
Q^E.K+CBF+BF & \text{constant} \\
\hline
\end{array}
\]

**LEMMA 7.** Every \( K \)-frame for \( Q^E.K+BF \) validates the necessity of fictionality, i.e., \( Q^E.K+BF \models N\neg E \).

We leave to the interested reader the proof of this standard result, which is due to decreasing inner domains. In the incompleteness result in section 4...
we will show that $Q^E.K+BF$ does not prove $N\neg E$. Lemma 7 applies also to the system $Q^E.K+CBF+BF$.

3 Counterpart Semantics

For introducing the counterpart semantics for QML we make use of typed languages. First, every variable $x_i$ in the alphabet $A$ is a term of type $n$, or $n$-term, for $n \geq i$. If $x_j$ is an $n$-term and $t_1, \ldots, t_n$ are $m$-terms, the substituted $m$-term $x_j[t_1, \ldots, t_n]$ is the $m$-term $t_j$, or $t_j : m$ in short.

DEFINITION 8. The typed first-order modal language $L_T$ contains all and only the formulas $\phi$ of type $n$, or $\phi : n$, for $n \in \mathbb{N}$, defined as follows:

- if $P^m$ is an $m$-ary predicative constant and $(t_1, \ldots, t_m)$ is an $m$-tuple of $n$-terms, then $P^m(t_1, \ldots, t_m)$ is a (atomic) formula of type $n$;
- if $\psi, \psi'$ are $n$-formulas, then $\neg \psi$ and $\psi \rightarrow \psi'$ are formulas of type $n$;
- if $\psi$ is an $m$-formula and $(t_1, \ldots, t_m)$ is an $m$-tuple of $n$-terms, then $(\Box \psi)(t_1, \ldots, t_m)$ is a formula of type $n$;
- if $\psi$ is an $n+1$-formula, then $\forall x_{n+1} \psi$ is a formula of type $n$.

The formula $\Box \psi : n$ is a shorthand for $(\Box \psi)(x_1, \ldots, x_n) : n$. Let $\phi$ be an $n$-formula and $\vec{s}$ an $n$-tuple of $k$-terms, the substituted $k$-formula $\phi[\vec{s}]$ is inductively defined as follows:

- $\phi$ is the atomic formula $P^m(t_1, \ldots, t_m)$, then $\phi[\vec{s}]$ is $P^m(t_1[\vec{s}], \ldots, t_m[\vec{s}])$;
- $\phi = \neg \psi$, then $\neg \psi[\vec{s}] = \neg(\psi[\vec{s}])$;
- $\phi = \psi \rightarrow \psi'$, then $(\psi \rightarrow \psi')[\vec{s}] = \psi[\vec{s}] \rightarrow \psi'[\vec{s}]$;
- $\phi = (\Box \psi)(t_1, \ldots, t_m)$, then $(\Box \psi)(t_1[\vec{s}], \ldots, t_m[\vec{s}]) = (\Box \psi)(t_1[\vec{s}], \ldots, t_m[\vec{s}])$;
- $\phi = \forall x_{n+1} \psi$, then $\forall x_{n+1} \psi[\vec{s}] = \forall x_{k+1}(\psi[\vec{s}, x_{k+1}])$.

Note that substitution does not commute with the modal operator, therefore it is not the case that $(\Box \psi)(t_1, \ldots, t_m)$ is equivalent to $\Box(\psi[t_1, \ldots, t_m])$.

DEFINITION 9. A counterpart frame, or $c$-frame, is a tuple $F = (W, R, D, d, C)$ such that $W$ is a non-empty set; $R \subseteq W^2$; for $w \in W$, $D(w)$ is a non-empty set and $d(w) \subseteq D(w)$; for $w R w'$, $C_{w, w'} \subseteq D(w) \times D(w')$.

In this paper we focus on the following classes of $c$-frames:

existentially faithful iff $w R w'$, $a \in d(w)$ and $C_{w, w'}(a, b)$, imply $b \in d(w')$

fictionally faithful iff $w R w'$, $a \in D(w) \setminus d(w)$ and $C_{w, w'}(a, b)$, imply $b \in D(w') \setminus d(w')$

everywhere-defined iff $w R w'$ and $a \in D(w)$, imply there is $b \in D(w')$ s.t. $C_{w, w'}(a, b)$

surjective iff $w R w'$ and $b \in d(w')$, imply there is $a \in d(w)$ s.t. $C_{w, w'}(a, b)$

functional iff $w R w'$, $C_{w, w'}(a, b)$ and $C_{w, w'}(a, b')$, imply $b = b'$

DEFINITION 10. A counterpart model for the language $L_T$ based on a $c$-frame $F$, or $c$-model in short, is a couple $M = (\mathcal{F}, \mathcal{I})$ where $\mathcal{I}$ is an interpretation of $L_T$ such that (i) if $P^m$ is an $n$-ary predicative constant and $w \in W$, then $I(P^m, w)$ is an $n$-ary relation on $D(w)$; (ii) $I(E, w) = d(w)$.

A finitary assignment of type $n$, or $n$-assignment, in a world $w$ is an $n$-tuple $\vec{a}$ of elements in $D(w)$. Let $t$ be the $n$-term $x_j$, the valuation $\vec{a}(t)$ for the $n$-assignment $\vec{a}$ is equal to $a_j$. 

DEFINITION 11. The satisfaction relation $\models$ for a world $w \in \mathcal{M}$, a typed formula $\phi : n$, and an $n$-assignment $\vec{a}$ is defined as follows:

- $(\mathcal{M}^{\vec{a}}, w) \models P^m(t_1, \ldots, t_m)$ if $(\vec{a}(t_1), \ldots, \vec{a}(t_m)) \in I(P^m, w)$
- $(\mathcal{M}^{\vec{a}}, w) \models \neg \psi$ if $(\mathcal{M}^{\vec{a}}, w) \not\models \psi$
- $(\mathcal{M}^{\vec{a}}, w) \models \psi \rightarrow \psi'$ if $(\mathcal{M}^{\vec{a}}, w) \not\models \psi$ or $(\mathcal{M}^{\vec{a}}, w) \models \psi'$
- $(\mathcal{M}^{\vec{a}}, w) \models (\Box \psi)(t_1, \ldots, t_m)$ if for $w' \in W$, for $b_1, \ldots, b_m \in D(w')$,
  $wRw'$ and $C_{w,w'}(\vec{a}(t_1), b_1) \implies (\mathcal{M}^{\vec{a}}, w') \models \psi$
- $(\mathcal{M}^{\vec{a}}, w) \models \forall x_{n+1} \psi$ if for every $a^* \in d(w)$, $(\mathcal{M}^{\vec{a}a^*}, w) \models \psi$

where $\vec{a} \cdot a^*$ is the $n + 1$-assignment $(a_1, \ldots, a_n, a^*)$.

A typed formula $\phi : n$ is said to be true at a world $w$ if it is satisfied by every $n$-assignment; $\phi$ is valid on a $c$-model $\mathcal{M}$ if it is true at every world in $\mathcal{M}$; $\phi$ is valid on a $c$-frame $\mathcal{F}$ if it is valid on every $c$-model based on $\mathcal{F}$.

4 Incompleteness of QML Systems

This section is devoted to the incompleteness proofs for systems $Q^E.K + BF$ and $Q^E.K + CBF + BF$, which are inspired to a similar result in [3]. We first show that $Q^E.K + BF$ is Kripke-incomplete, that is, there is no class of Kripke frames which validates all and only the theorems of $Q^E.K + BF$.

THEOREM 12. The system $Q^E.K + BF$ is Kripke-incomplete, i.e., every $K$-frame for $Q^E.K + BF$ validates $\neg \neg E$, but $Q^E.K + BF \not\vdash \neg \neg E$.

In section 2 we remarked that $Q^E.K + BF \not\models \neg \neg E$. In order to show that $Q^E.K + BF$ does not prove $\neg \neg E$ we need two lemmas. By the first one if a formula $\phi \in \mathcal{L}$ is a theorem in $Q^E.K + BF$, then its translation $\tau_n(\phi) \in \mathcal{L}_T$ as defined below holds in a suitable $c$-frame. By the second lemma this suitable $c$-frame does not validate $\neg \neg E(x_n) \rightarrow \Box \neg \neg E(x_n)$, i.e., the translation of $\neg \neg E$ according to $\tau_n$. By contraposition we obtain that $Q^E.K + BF$ does not prove $\neg \neg E$.

Following [3, 6] we define a translation function from untyped to typed first-order modal languages.

DEFINITION 13. Let $\phi \in \mathcal{L}$ be an untyped formula and define $g(\phi)$ as the maximum $k$ such that $x_k$ occurs in $\phi$. For $n \geq g(\phi)$, the formula $\tau_n(\phi) \in \mathcal{L}_T$ of type $n$ is inductively defined as follows:

- $\tau_n(P^m(t_1, \ldots, t_m)) := P^m(t_1, \ldots, t_m)$
- $\tau_n(\neg \psi) := \neg \tau_n(\psi)$
- $\tau_n(\Box \psi) := \Box \tau_n(\psi)$
- $\tau_n(\forall x_{n+1} \psi) := \forall x_{n+1} \tau_n(\psi)[x_1, \ldots, x_{n+1}, x_{n+1}, \ldots, x_n]$

By the first lemma theoremhood in $Q^E.K + BF$ implies validity in everywhere-defined, surjective, functional $c$-frames, modulo the translation function $\tau_n$.

LEMMA 14. Let $\phi \in \mathcal{L}$, $n \geq g(\phi)$ and let $\mathcal{F}$ be an everywhere-defined, surjective, and functional $c$-frame, then

$Q^E.K + BF \vdash \phi$ implies $\mathcal{F} \models \tau_n(\phi)$
The proof of this lemma requires the following auxiliary result, in which the assumptions of everywhere-definiteness and functionality are essential.

**LEMMA 15.** If \( \phi \) is a formula in \( \mathcal{L} \), \( \mathcal{F} \) is an everywhere-defined and functional c-frame, and \( x_{i_1}, \ldots, x_{i_m} \) are free for \( x_1, \ldots, x_m \) in \( \phi \), then

\[
\mathcal{F} \models \tau_m(\phi)[x_{i_1}, \ldots, x_{i_m}] \iff \tau_n(\phi[x_{i_1}, \ldots, x_{i_m}])
\]

If \( Q^E.K+BF \) proves \( \neg E \), then any everywhere-defined, surjective, and functional c-frame models \( \tau_n(\neg E) \). But the latter fact is negated by the next lemma.

**LEMMA 16.** There exists an everywhere-defined, surjective, and functional c-frame \( \mathcal{F} \) such that \( \mathcal{F} \not\models \neg E(x_n) \rightarrow \Box \neg E(x_n) : n \).

**Proof.** Consider the c-frame \( \mathcal{F} \), where \( W = \{w, w'\} \); \( R = \{(w, w')\} \); \( D(w) = \{a, a'\}, D(w') = \{b\}; d(w) = \{a\}, d(w') = \{b\}; C_{w,w'} = \{(a, b), (a', b)\} \).

By definition \( \mathcal{F} \) is everywhere-defined, surjective, and functional, but \( \neg E \) fails in \( \mathcal{F} \) as it is not fictionally faithful. Consider a c-model \( \mathcal{M} \) based on \( \mathcal{F} \) and an n-assignment \( \vec{a} \) such that \( a_n = a' \) and \( (\mathcal{M}^E, w) \models \neg E(x_n) \).

We have that \( C_{w,w'}(a', b) \) and \( b \in d(w') \), so \( (\mathcal{M}^E, w) \models \Box E(x_n) \). Thus, \( (\mathcal{M}^E, w) \models \neg E(x_n) \wedge \Box E(x_n) \) and \( \mathcal{F} \not\models \neg E \).

By lemmas 14 and 16 the system \( Q^E.K+BF \) does not prove \( \neg E \), which is nonetheless valid on every \( K \)-frame for \( Q^E.K+BF \). As a result, theorem 12 holds.

Note that also the system \( Q^E.K+CBF+BF \) is Kripke-incomplete, as lemma 14 holds also for \( Q^E.K+CBF+BF \) with respect to existentially faithful, everywhere-defined, surjective, and functional c-frames. Further, the c-frame in lemma 16 is also existentially faithful.

**THEOREM 17.** The system \( Q^E.K+CBF+BF \) is Kripke-incomplete, i.e., \( Q^E.K+CBF+BF \not\models N \neg E \), but \( Q^E.K+CBF+BF \not\models Q^E.K+CBF \not\models \neg E \).

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**BIBLIOGRAPHY**


