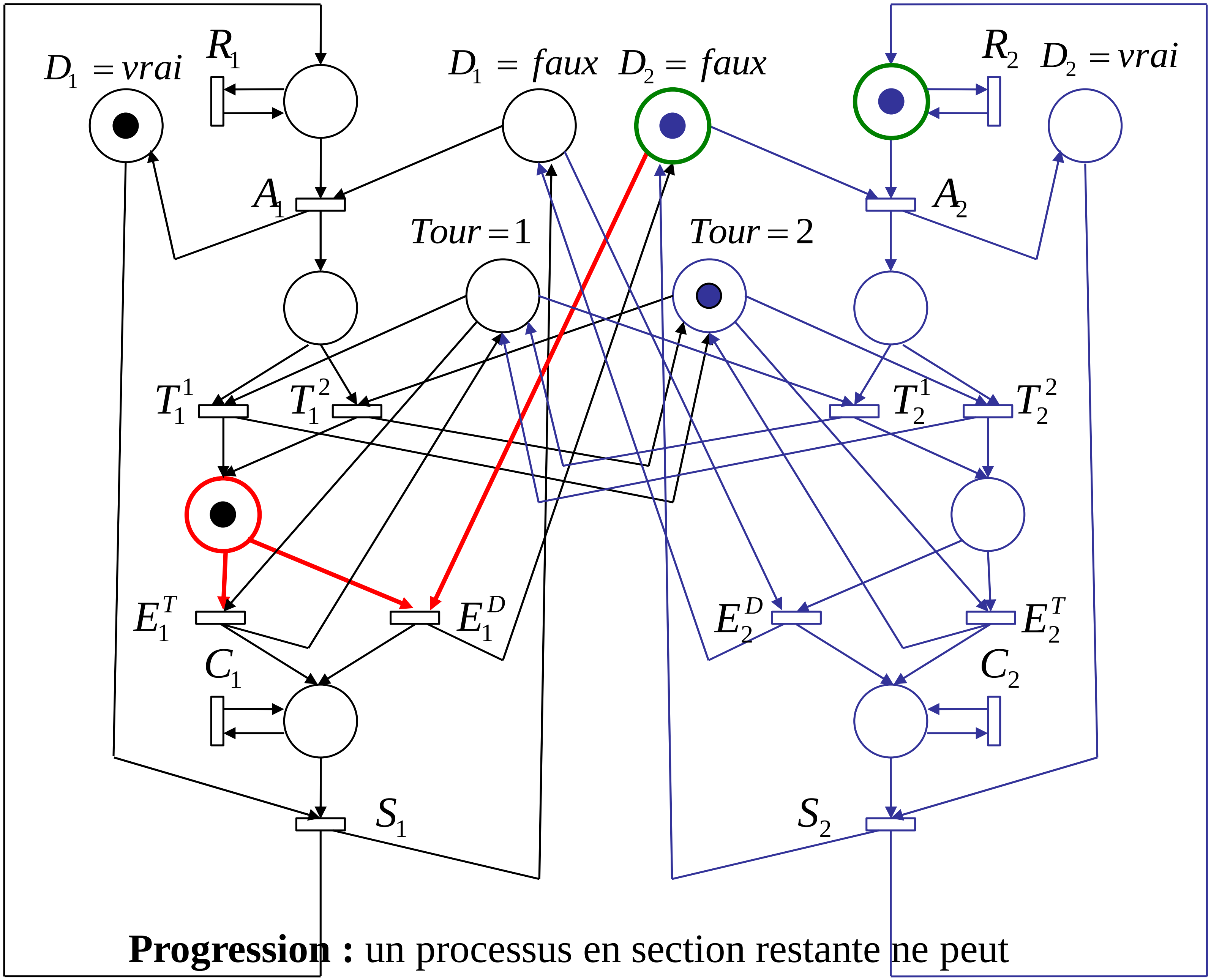
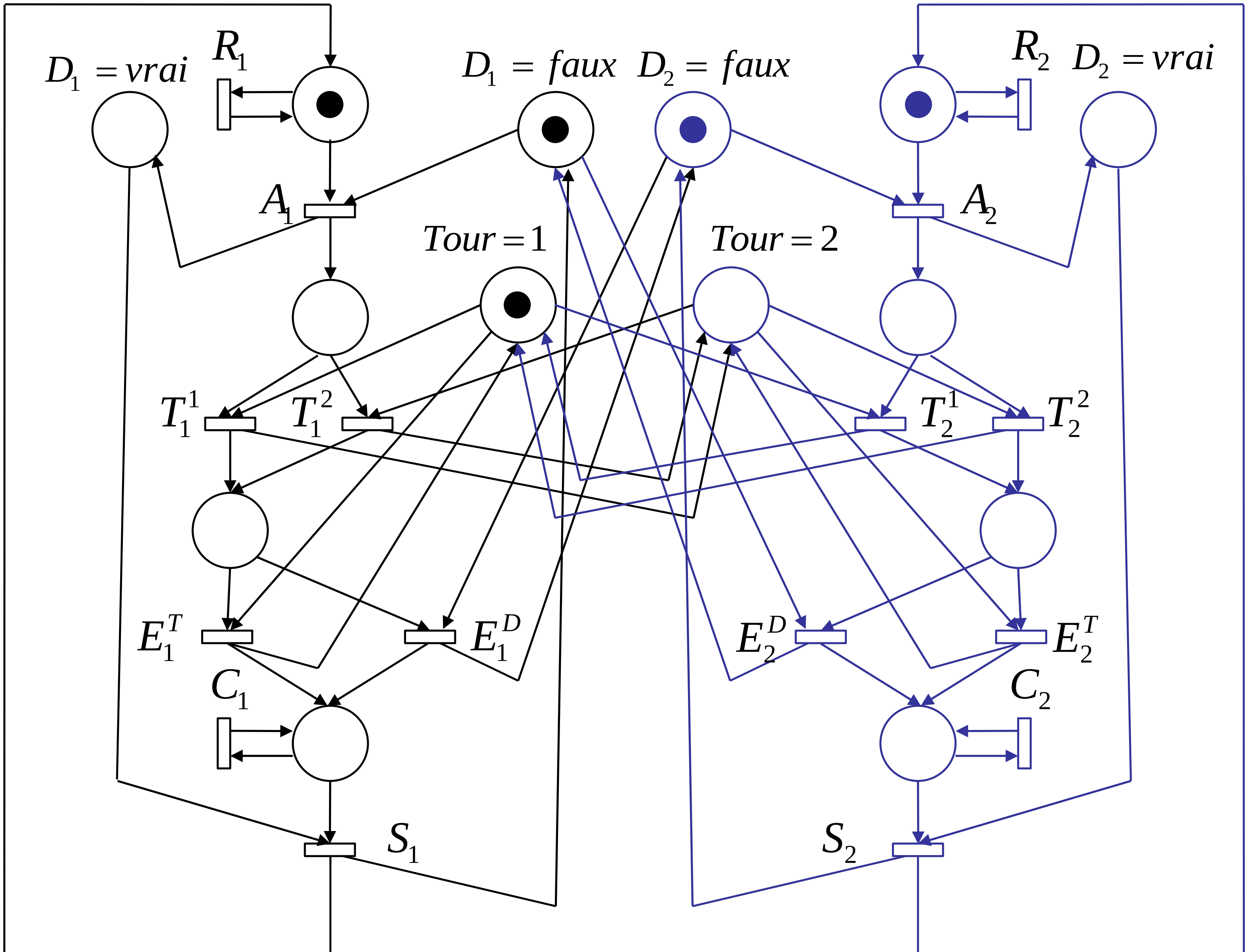
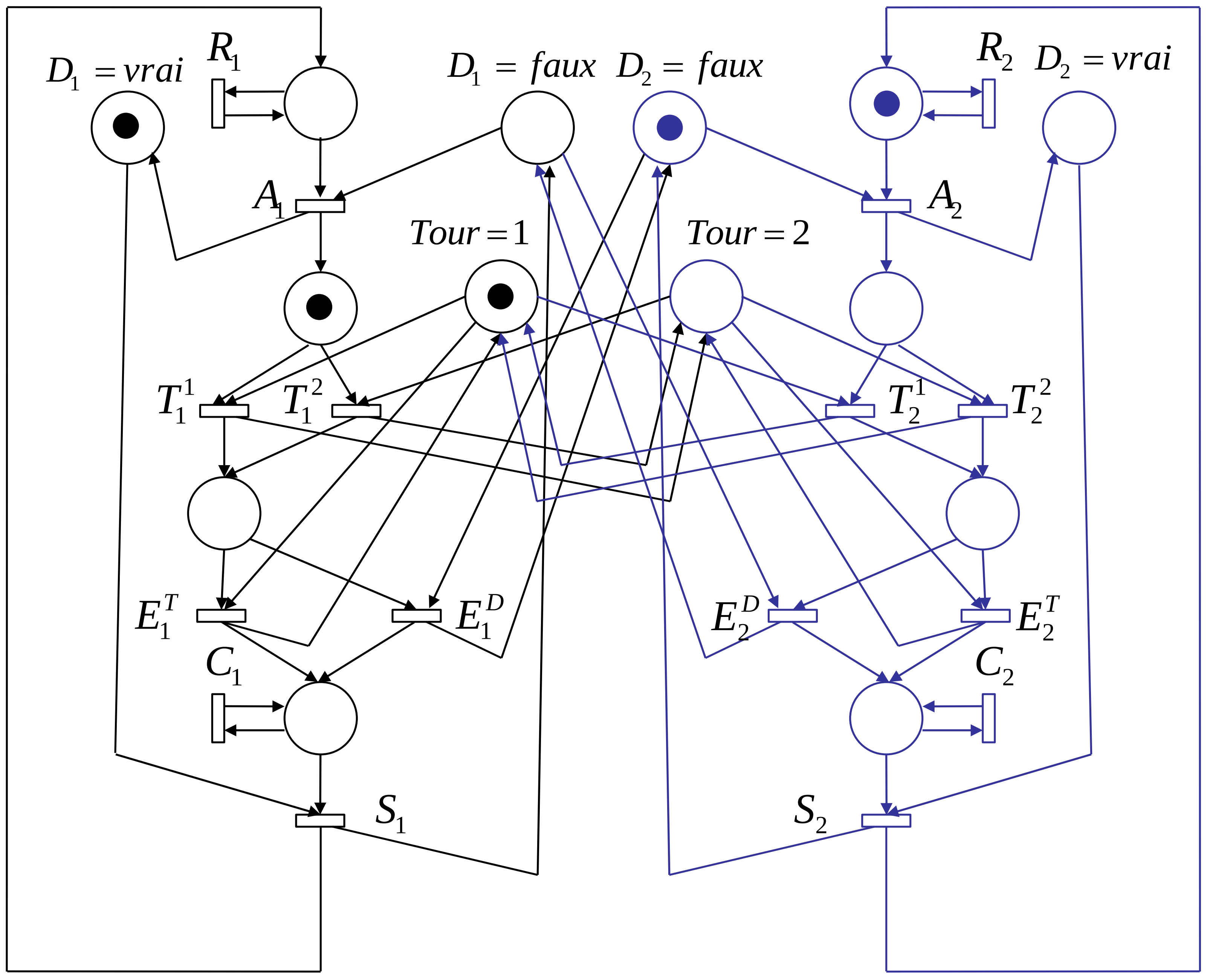



**Progression :** un processus en section restante ne peut empêcher l'autre processus de rentrer en section critique.

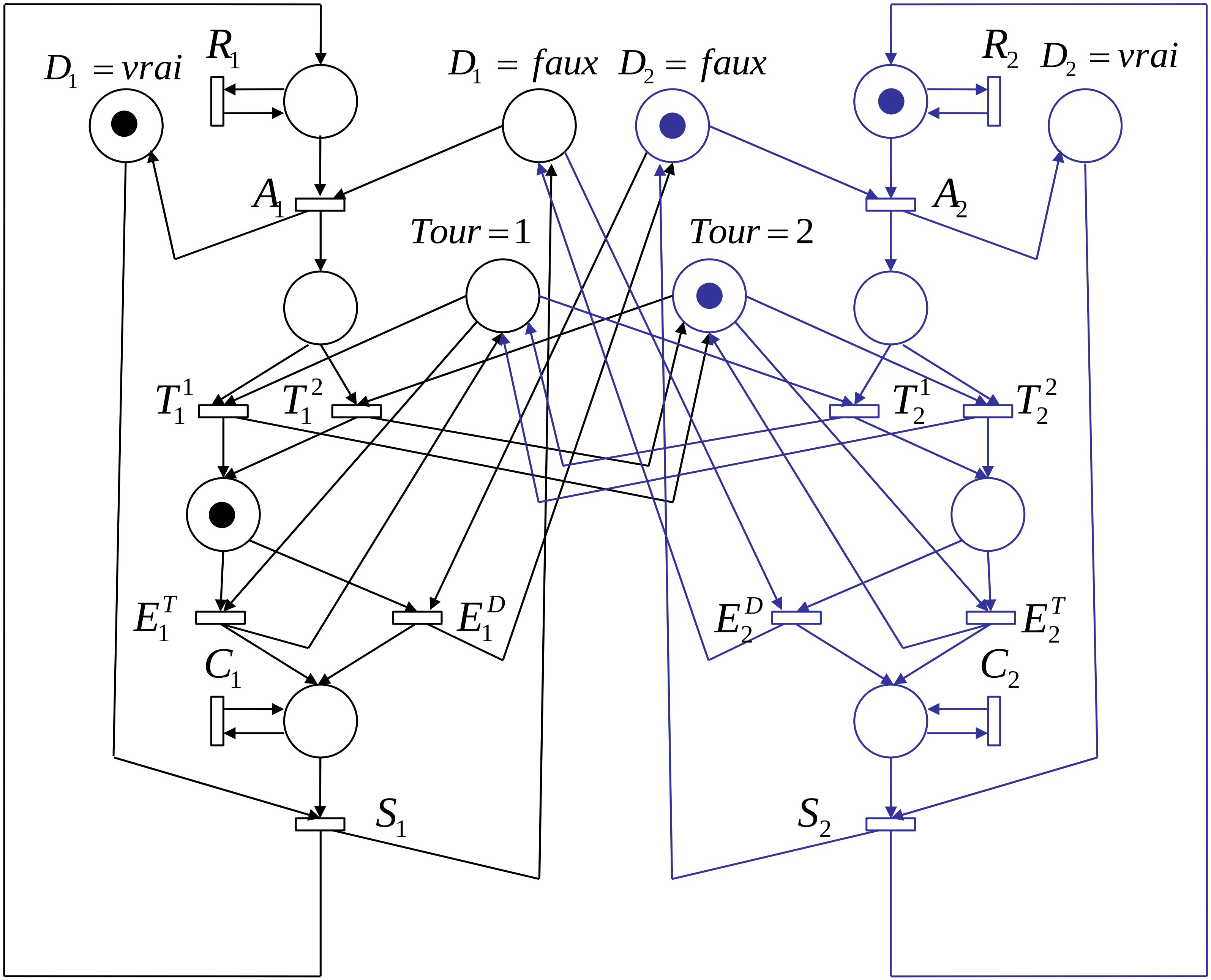
J.-M. Delosme - SED



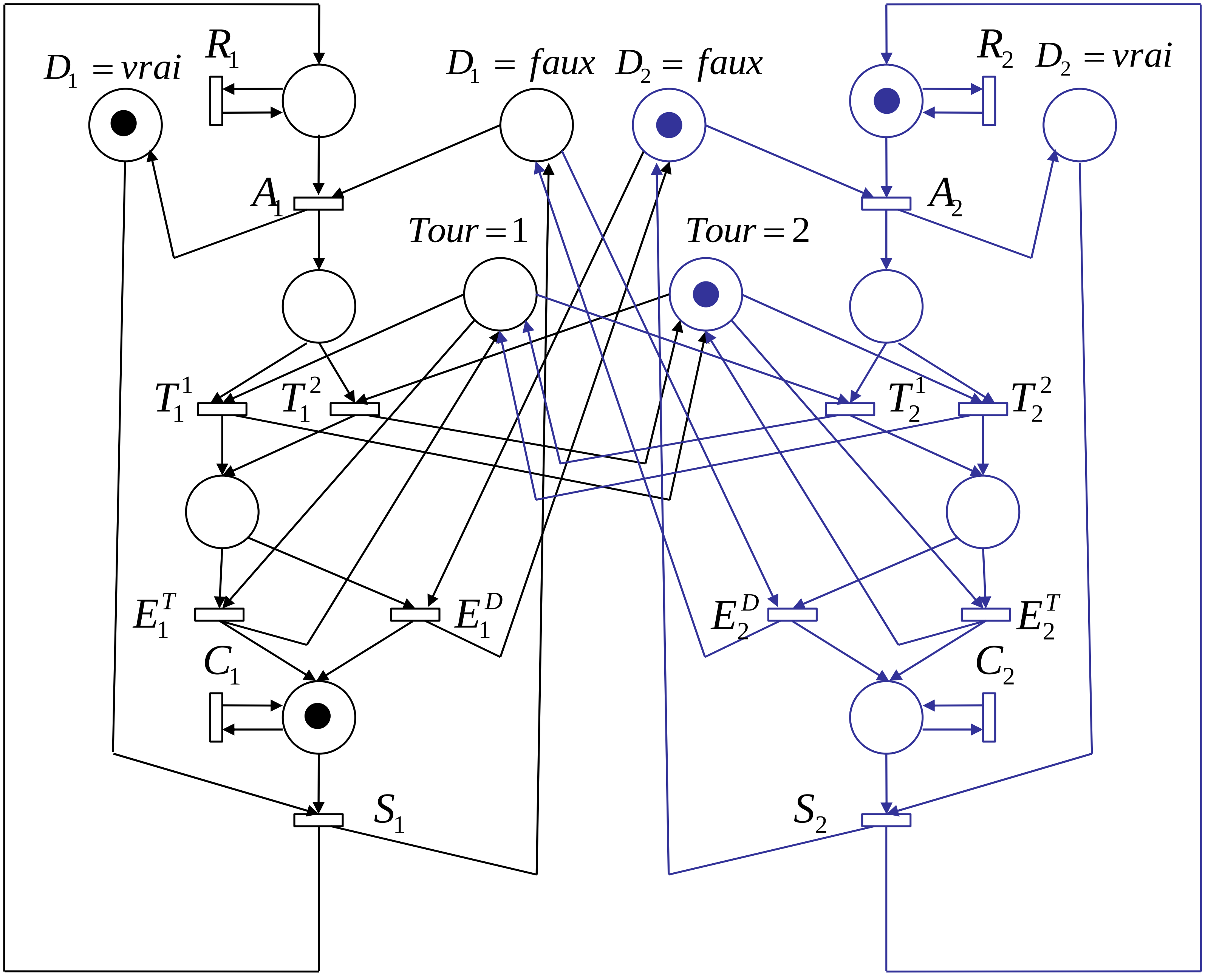

Dans l'état initial, $P_2$ est en section restante (et $Tour = 1$).
On essaye donc de faire progresser $P_1$.



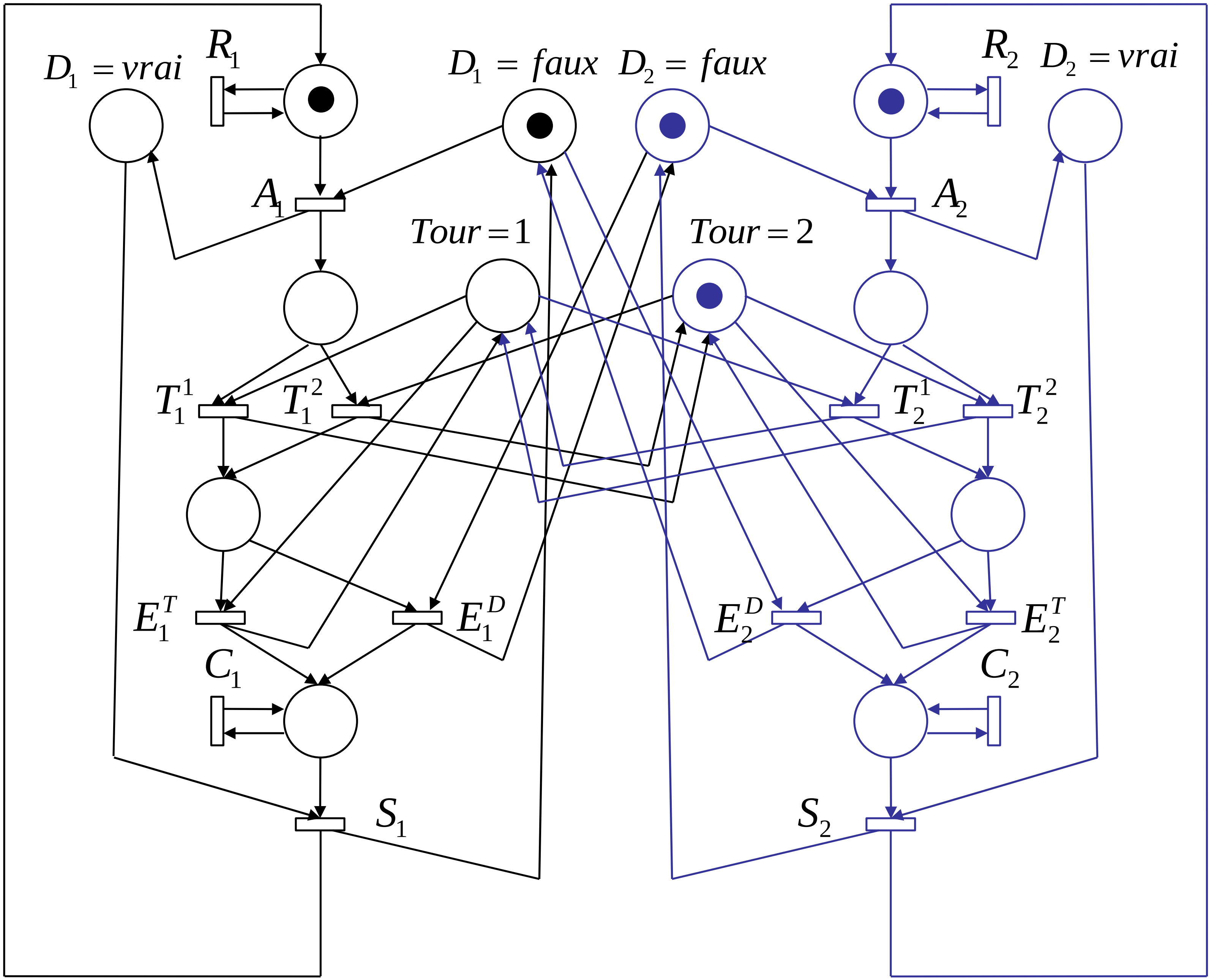

J.-M. Delosme - SED



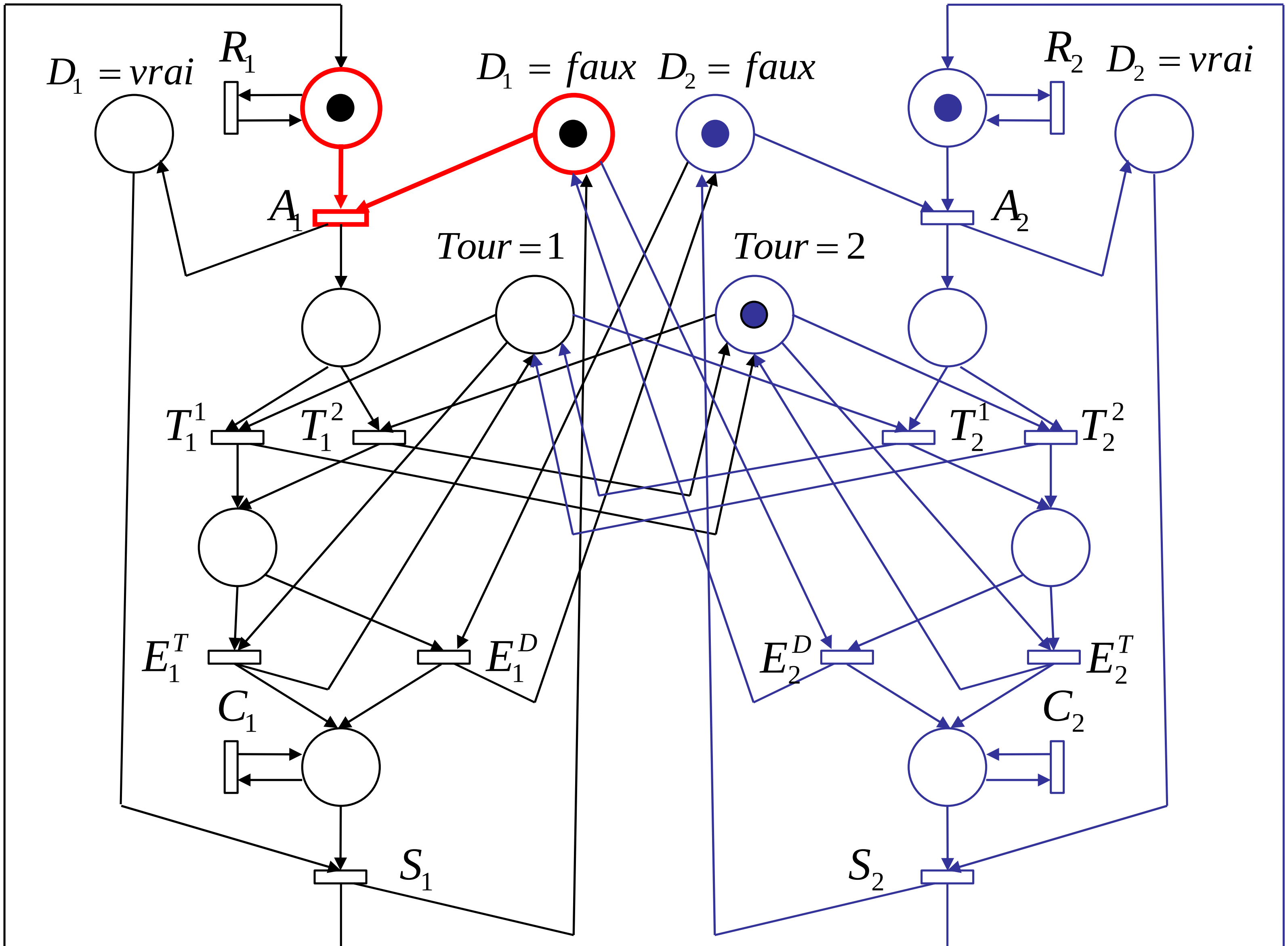

J.-M. Delosme - SED



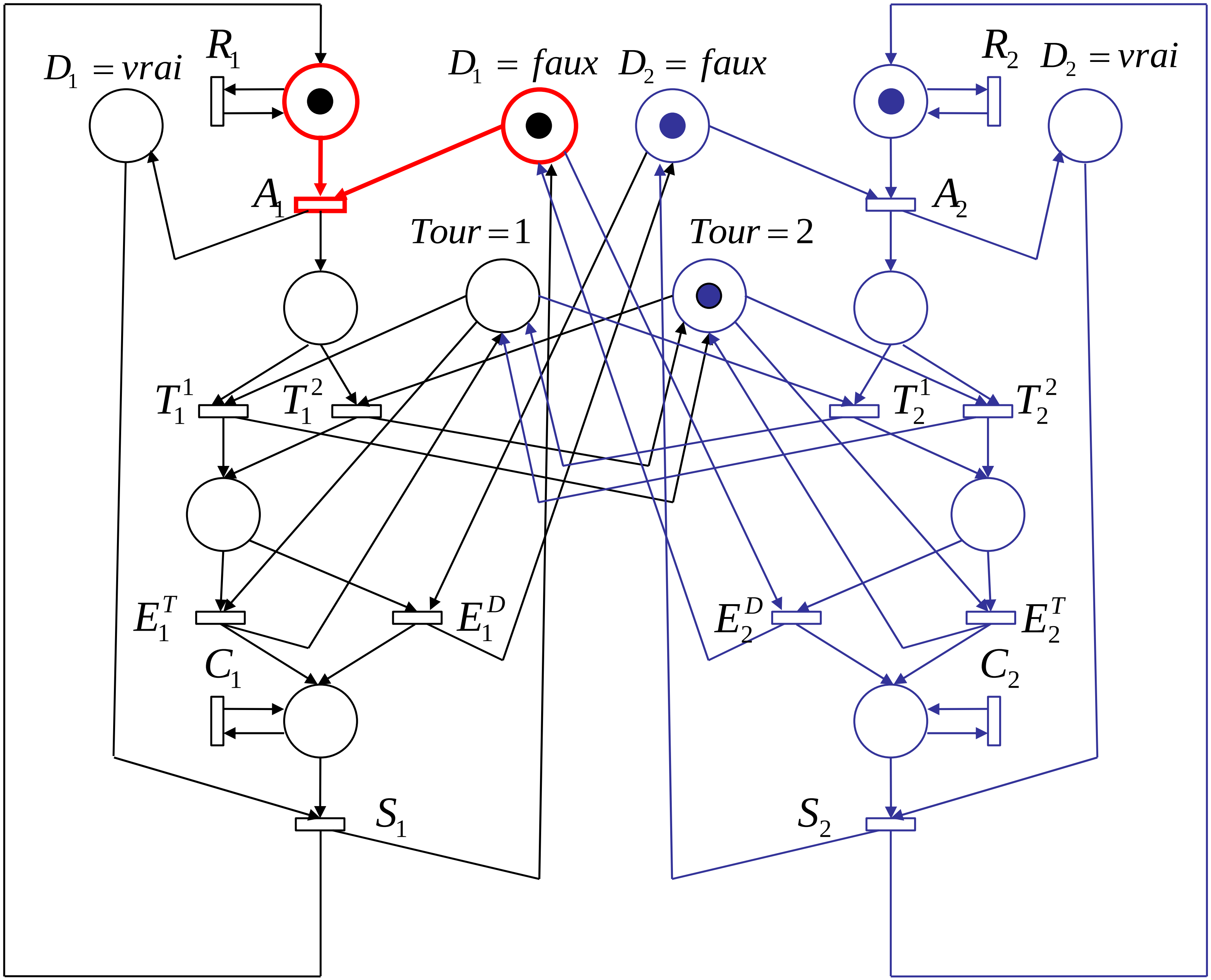

J.-M. Delosme - SED



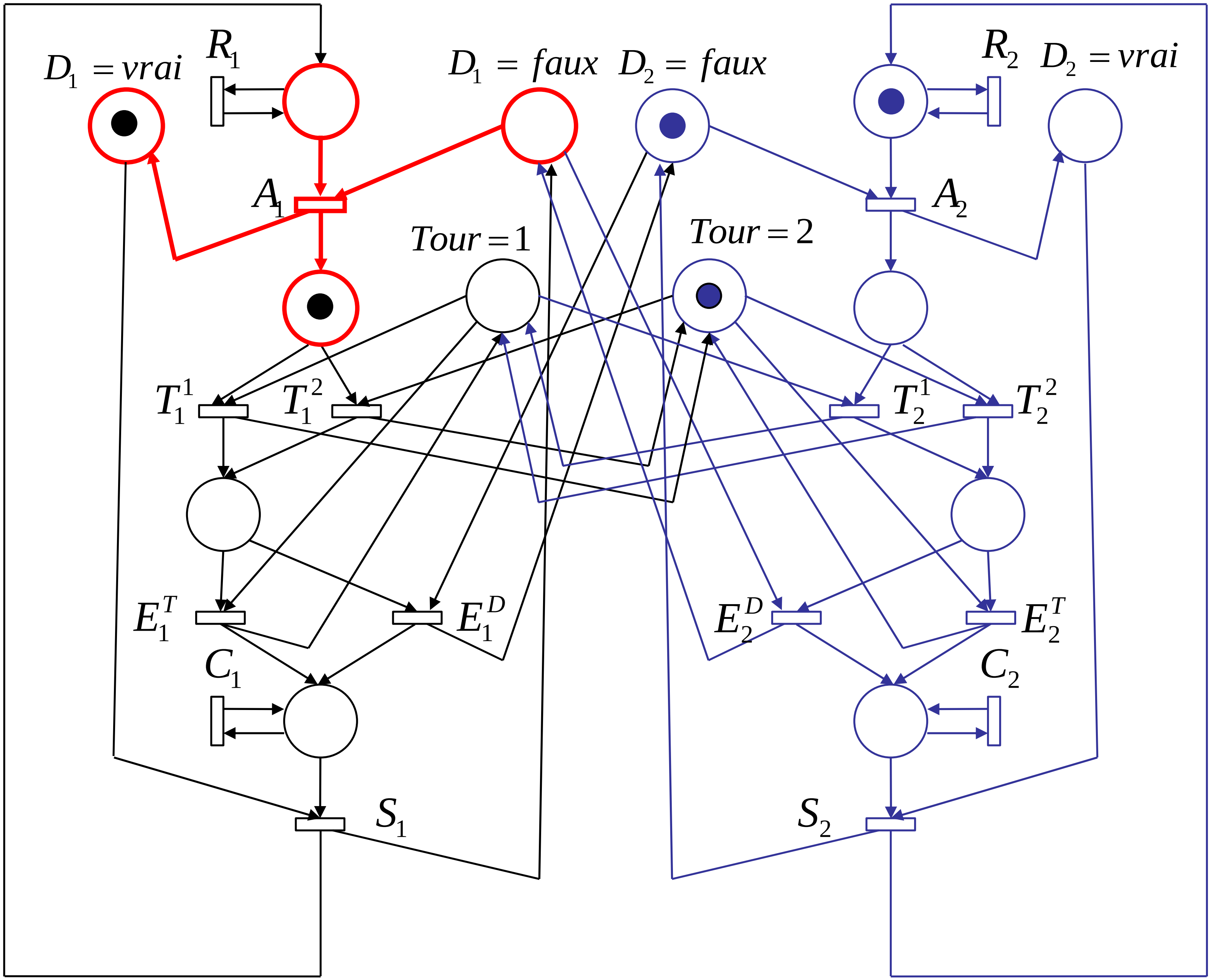

J.-M. Delosme - SED



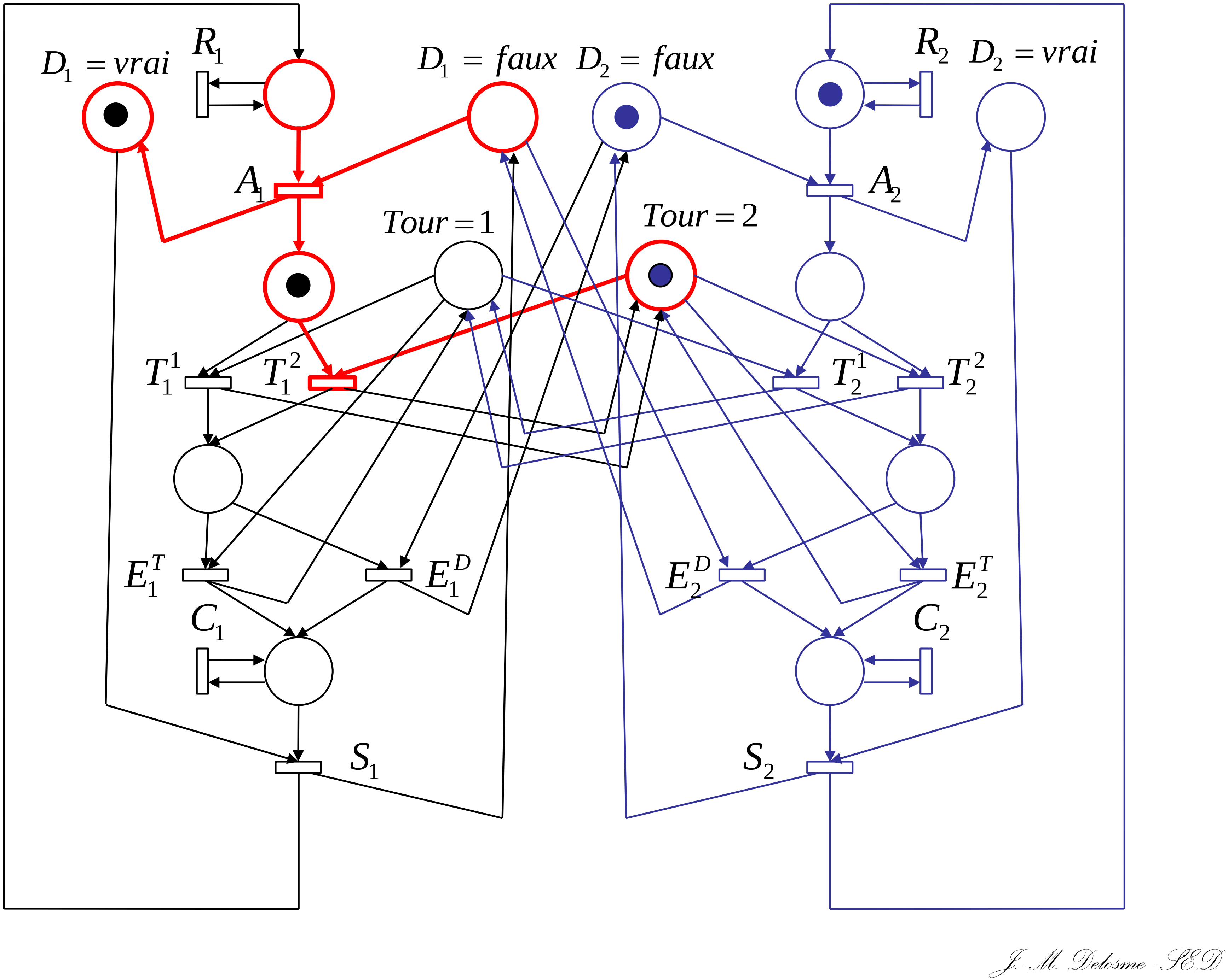

Le processus $P_1$ a pu faire une itération (et maintenant $Tour = 2$). Pour prouver la progression (nombre d'itérations de $P_1$ non borné) on utilise la notion de T-invariant.



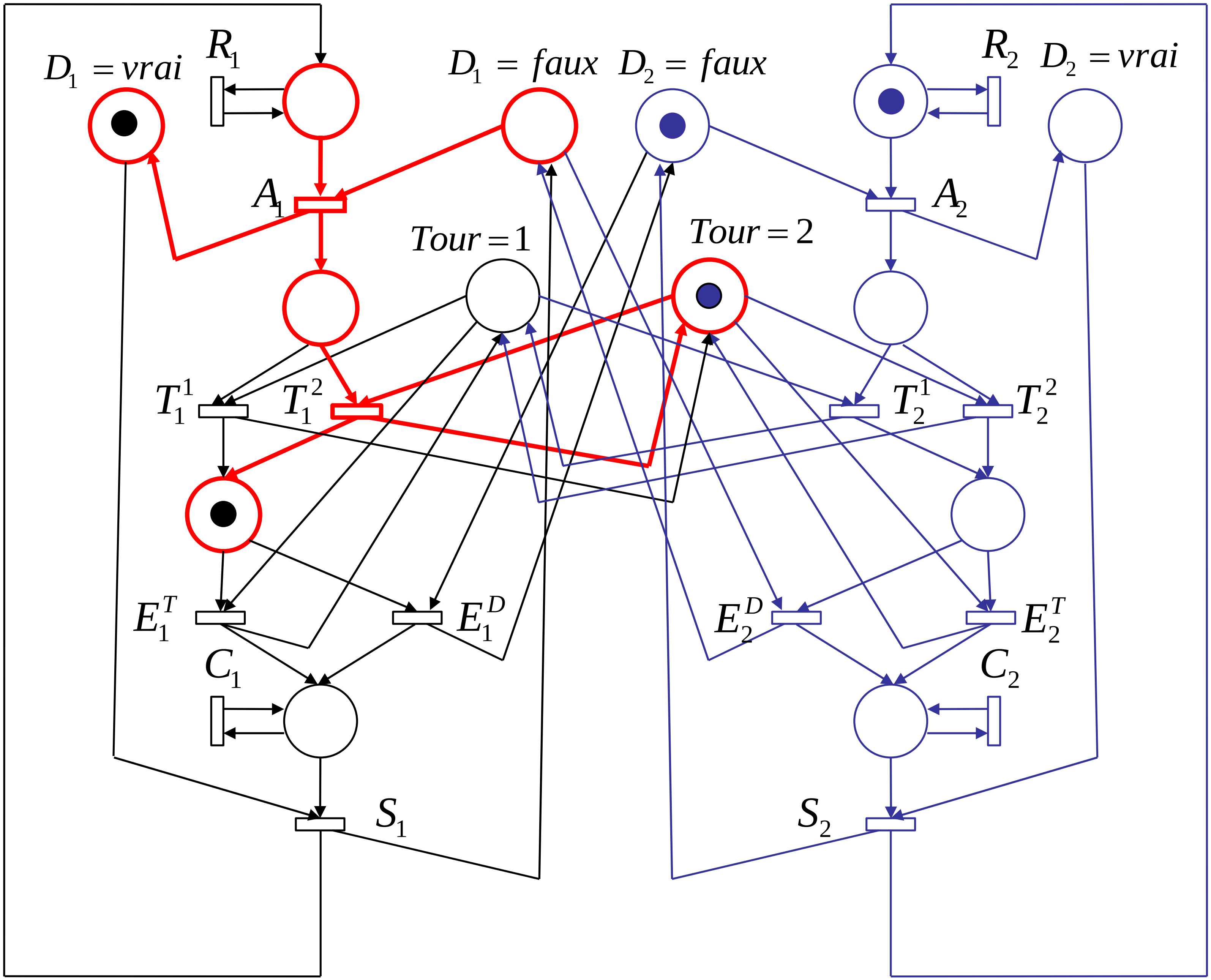

J.-M. Delosme - SED



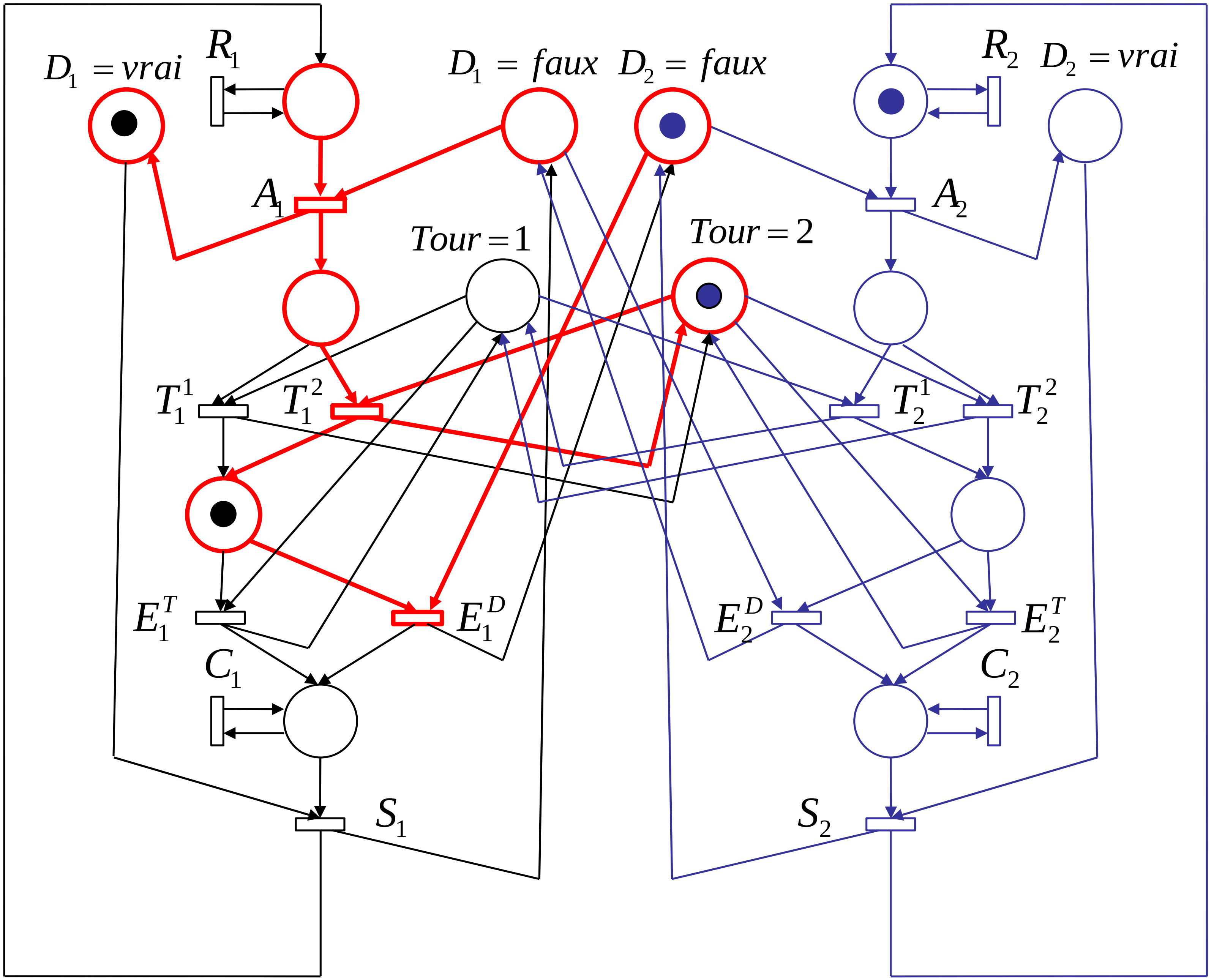

J.-M. Delosme - SED



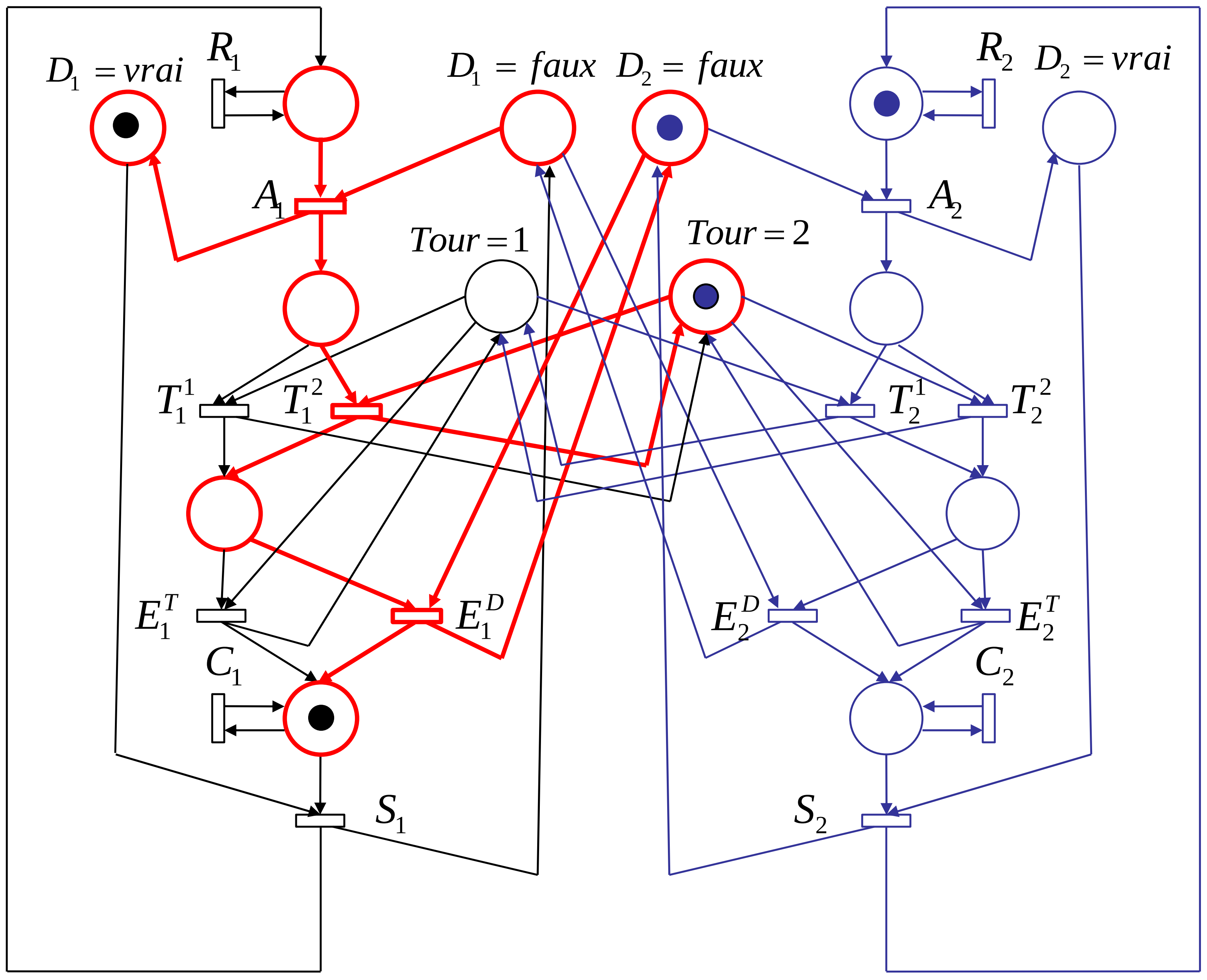
$D_1 = vrai$
$R_1$
$D_1 = faux$
$D_2 = faux$
$R_2$
$D_2 = vrai$
$A_1$
$A_2$
$Tour = 1$
$Tour = 2$
$T_1^1$
$T_1^2$
$T_2^1$
$T_2^2$
$E_1^T$
$E_1^D$
$E_2^D$
$E_2^T$
$C_1$
$C_2$
$S_1$
$S_2$
J.-M. Delosme - SED



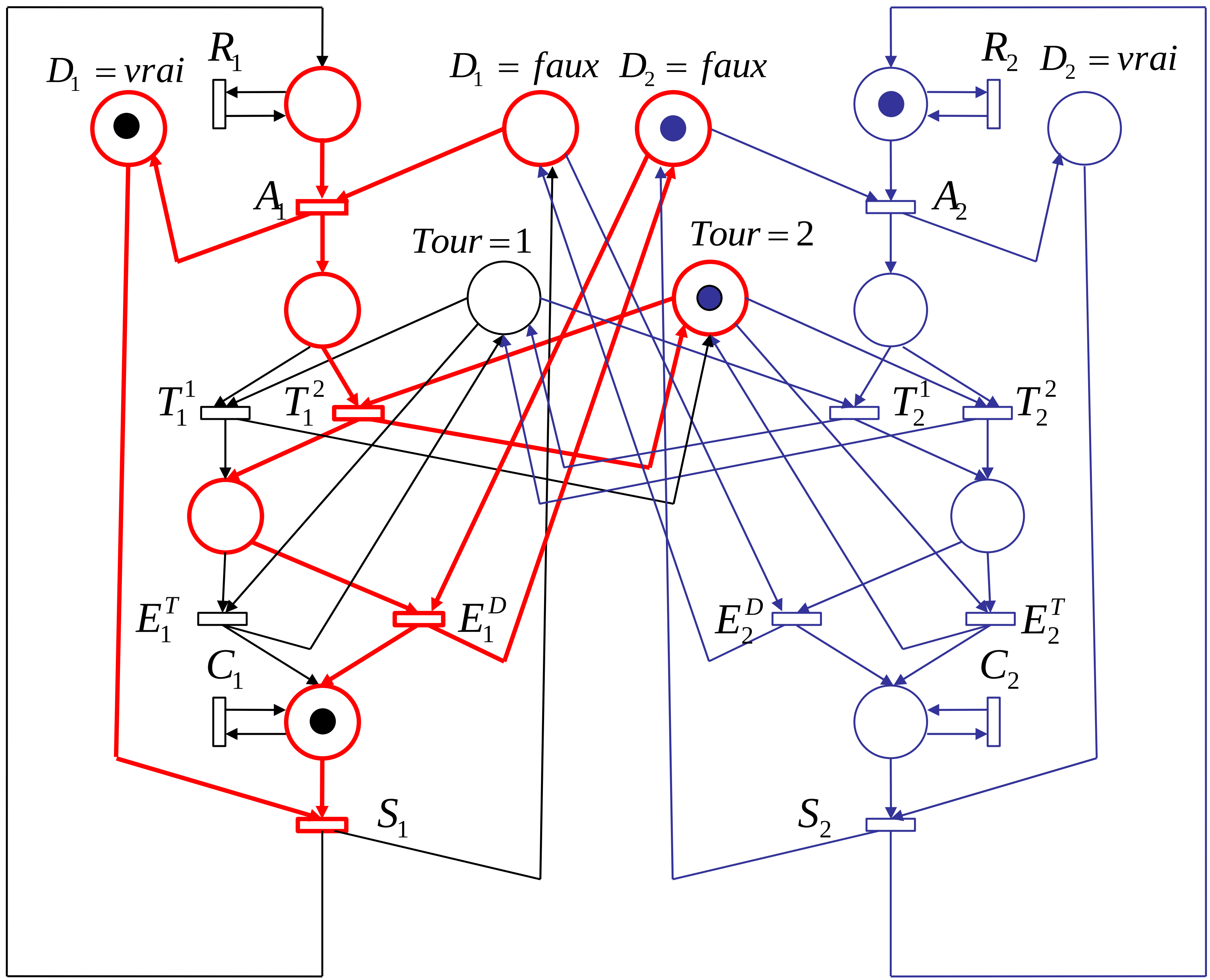

J.-M. Delosme - SED



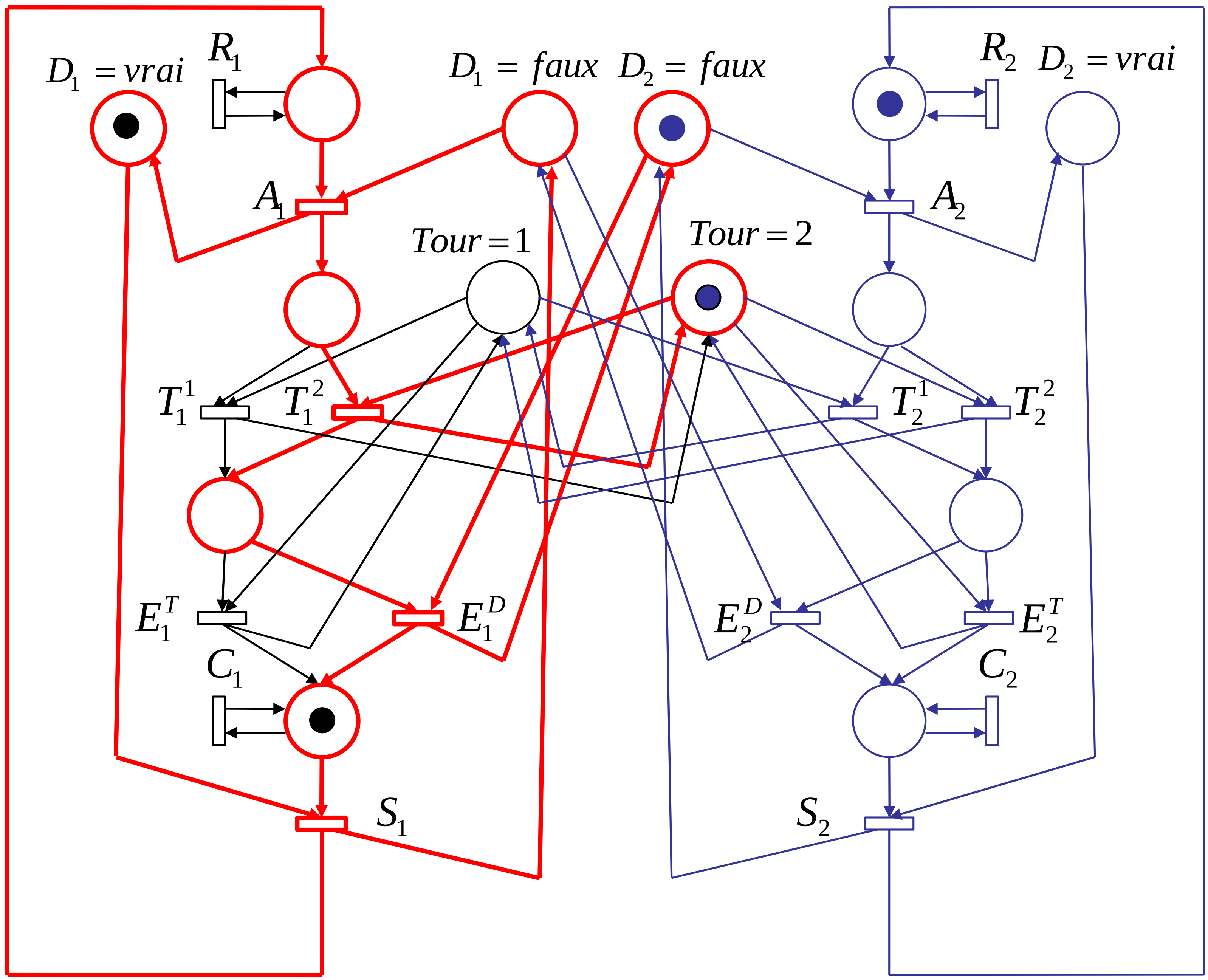

J.-M. Delosme - SED



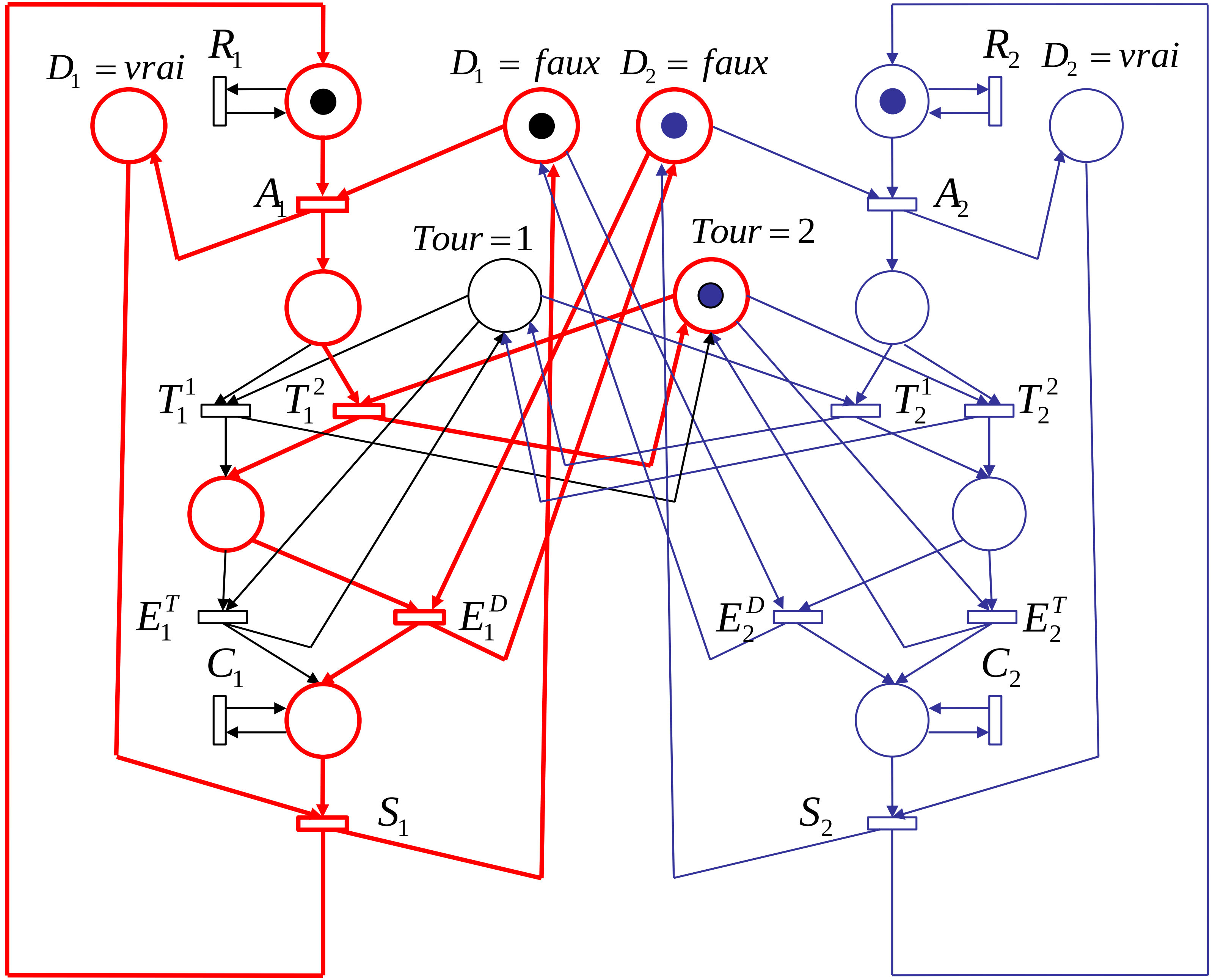

J.-M. Delosme - SED




J.-M. Delosme - SED

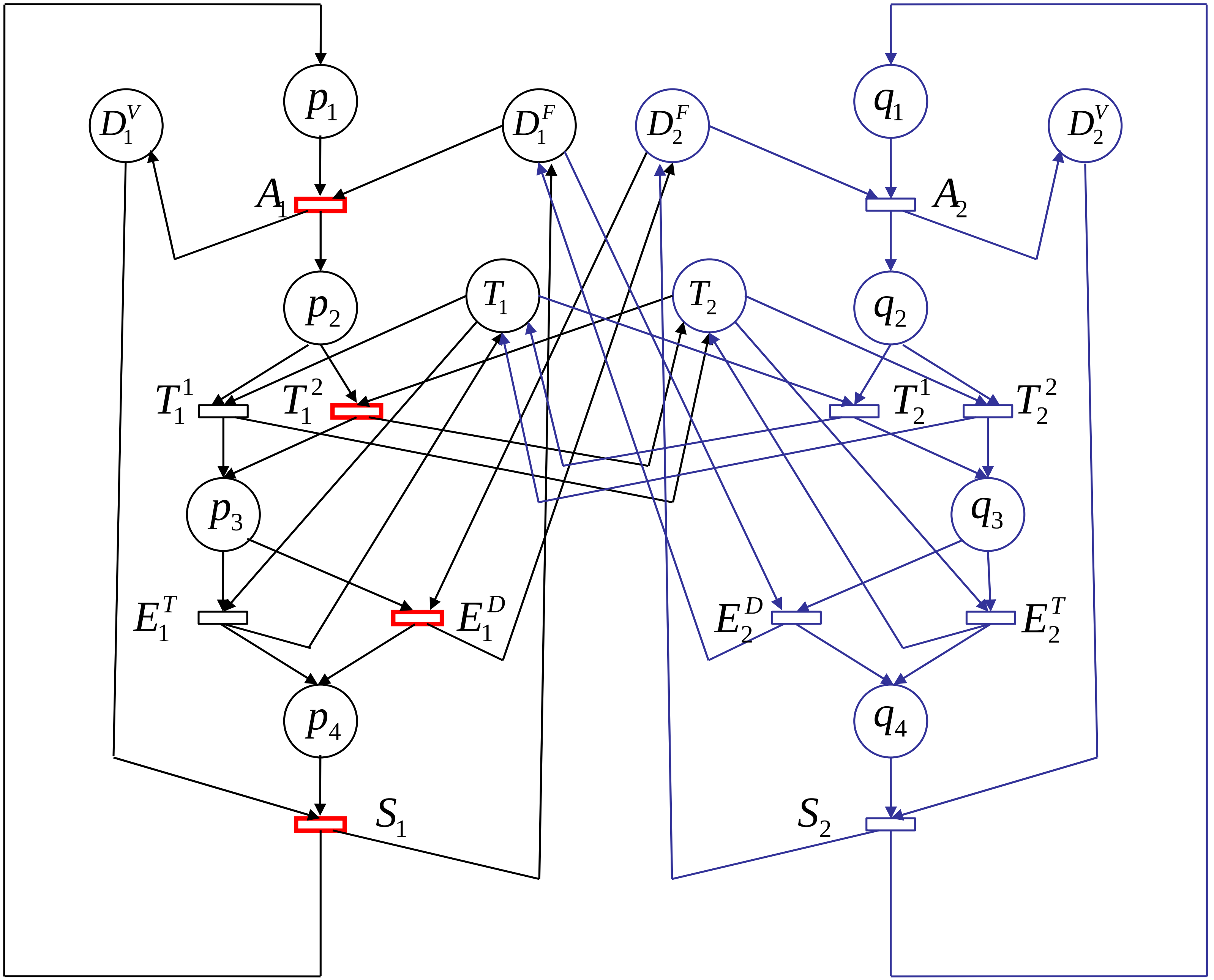



J.-M. Delosme - SED




T-invariant : retour à l'état initial (avec $Tour = 2$).

J.-M. Delosme - SED




J.-M. Delosme - SED

| | $D_1^F$ | $D_1^V$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $T_1$ | $T_2$ | $D_2^F$ | $D_2^V$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | −1 | 1 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^1$ | 0 | 0 | 0 | −1 | 1 | 0 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^2$ | 0 | 0 | 0 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^T$ | 0 | 0 | 0 | 0 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^D$ | 0 | 0 | 0 | 0 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $S_1$ | 1 | −1 | 1 | 0 | 0 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $A_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | −1 | 1 | 0 | 0 |
| $T_2^1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | 0 |
| $T_2^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 0 | 0 | 0 | −1 | 1 | 0 |
| $E_2^T$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 |
| $E_2^D$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 |
| $S_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 1 | 0 | 0 | −1 |

T-invariant

$$\boldsymbol{u} = [1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0]$$

$$\boldsymbol{u}B = 0 \Rightarrow \boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}B = \boldsymbol{x}$$

| | $D_1^F$ | $D_1^V$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $T_1$ | $T_2$ | $D_2^F$ | $D_2^V$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | –1 | 1 | –1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^1$ | 0 | 0 | 0 | –1 | 1 | 0 | –1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^2$ | 0 | 0 | 0 | –1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^T$ | 0 | 0 | 0 | 0 | –1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^D$ | 0 | 0 | 0 | 0 | –1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $S_1$ | 1 | –1 | 1 | 0 | 0 | –1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $A_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | –1 | 1 | –1 | 1 | 0 | 0 |
| $T_2^1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | –1 | 1 | 0 |
| $T_2^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | –1 | 0 | 0 | 0 | –1 | 1 | 0 |
| $E_2^T$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | –1 | 1 |
| $E_2^D$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | –1 | 1 |
| $S_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | –1 | 1 | 0 | 0 | –1 |

Pour calculer les T-invariants on commence par exploiter le résultat du calcul des P-invariants et on élimine 7 colonnes linéairement dépendantes de celles conservées.

$$
\begin{array}{c}
A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2
\end{array}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

Pour mieux voir les dépendances entre lignes on réduit la matrice sous forme échelonnée par des opérations sur les colonnes.

+1

$$
\begin{array}{c}
A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2
\end{array}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

$$
\begin{array}{c}
A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2
\end{array}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

$$
\begin{array}{c}
A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2
\end{array}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

Forme échelonnée

$$
+\;
\begin{matrix} A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2 \end{matrix}
\begin{bmatrix}
\boxed{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
\boxed{1} & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

$$
\begin{array}{c}
A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2
\end{array}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

+

$$
\begin{array}{c}
A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2
\end{array}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

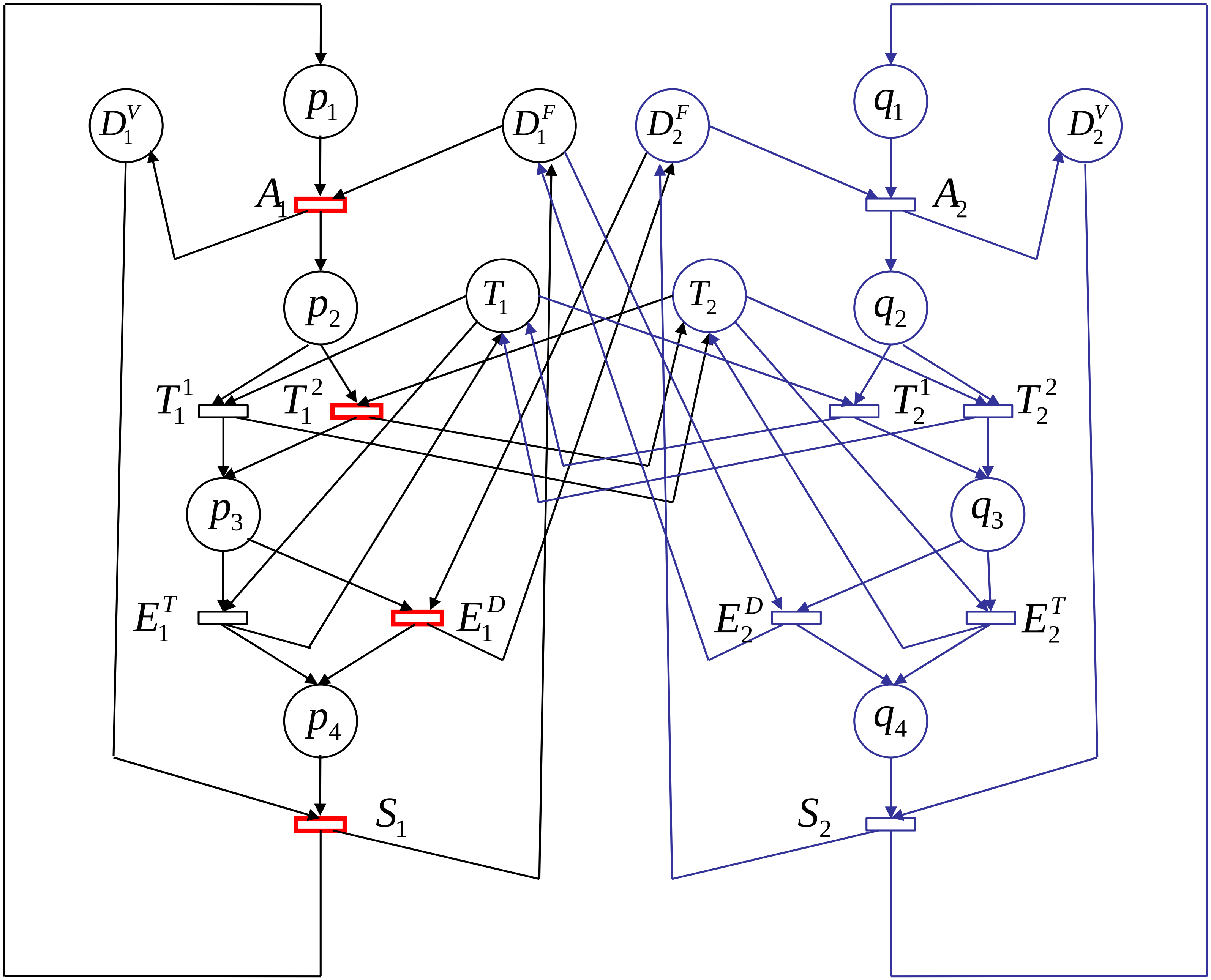
$$
\begin{matrix} A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2 \end{matrix}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

$$
+\quad
\begin{array}{r}
A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2
\end{array}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

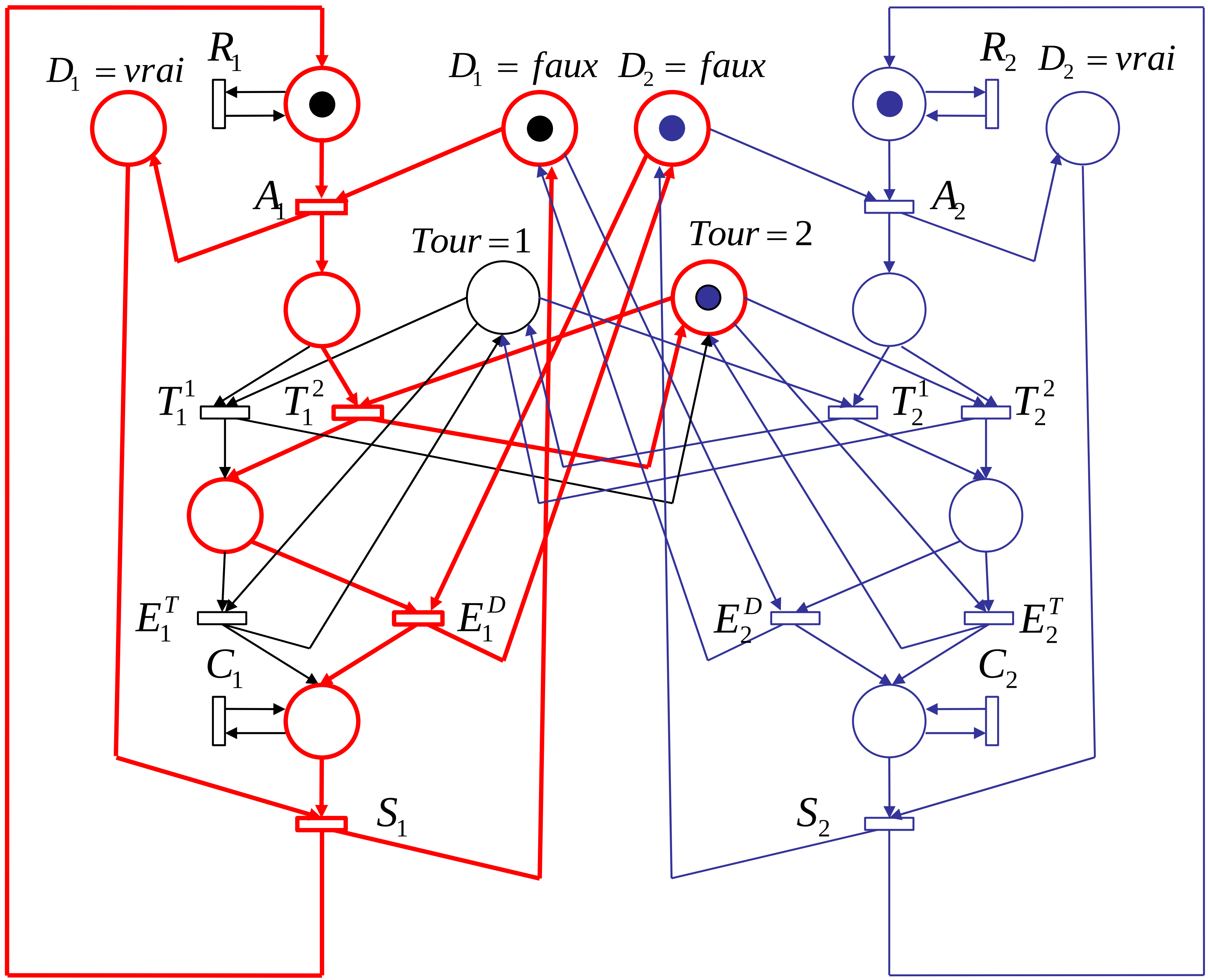
| | $D_1^F$ | $D_1^V$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $T_1$ | $T_2$ | $D_2^F$ | $D_2^V$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | −1 | 1 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^1$ | 0 | 0 | 0 | −1 | 1 | 0 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^2$ | 0 | 0 | 0 | −1 | 1 | 0 | 0 | **0** | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^T$ | 0 | 0 | 0 | 0 | −1 | 1 | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^D$ | 0 | 0 | 0 | 0 | −1 | 1 | 0 | 0 | **0** | 0 | 0 | 0 | 0 | 0 |
| $S_1$ | 1 | −1 | 1 | 0 | 0 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $A_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | −1 | 1 | 0 | 0 |
| $T_2^1$ | 0 | 0 | 0 | 0 | 0 | 0 | **0** | 0 | 0 | 0 | 0 | −1 | 1 | 0 |
| $T_2^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 0 | 0 | 0 | −1 | 1 | 0 |
| $E_2^T$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **0** | 0 | 0 | 0 | 0 | −1 | 1 |
| $E_2^D$ | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 |
| $S_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 1 | 0 | 0 | −1 |

T-invariant 1

$$\boldsymbol{u}_1 = [1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0]$$

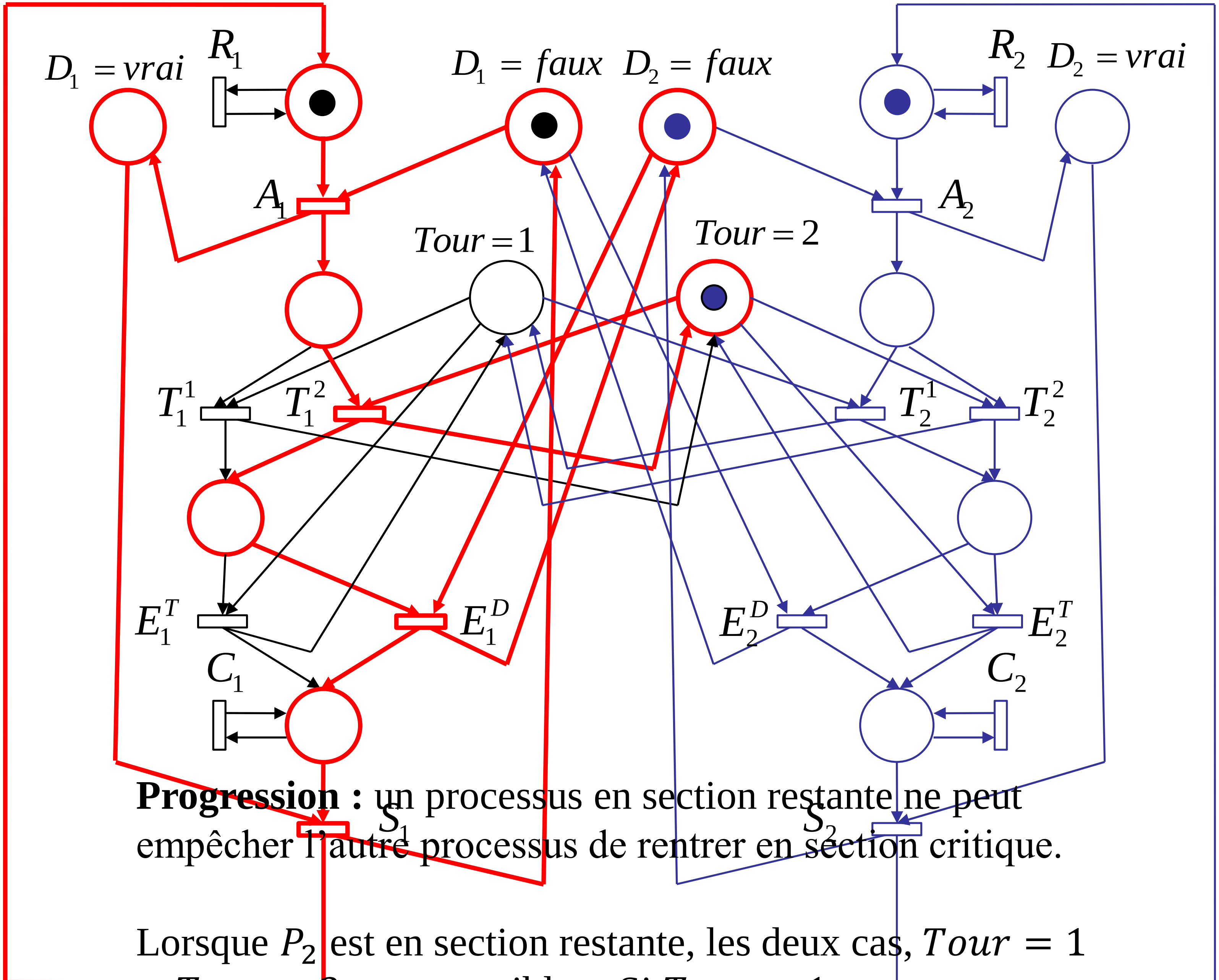



J.-M. Delosme - SED



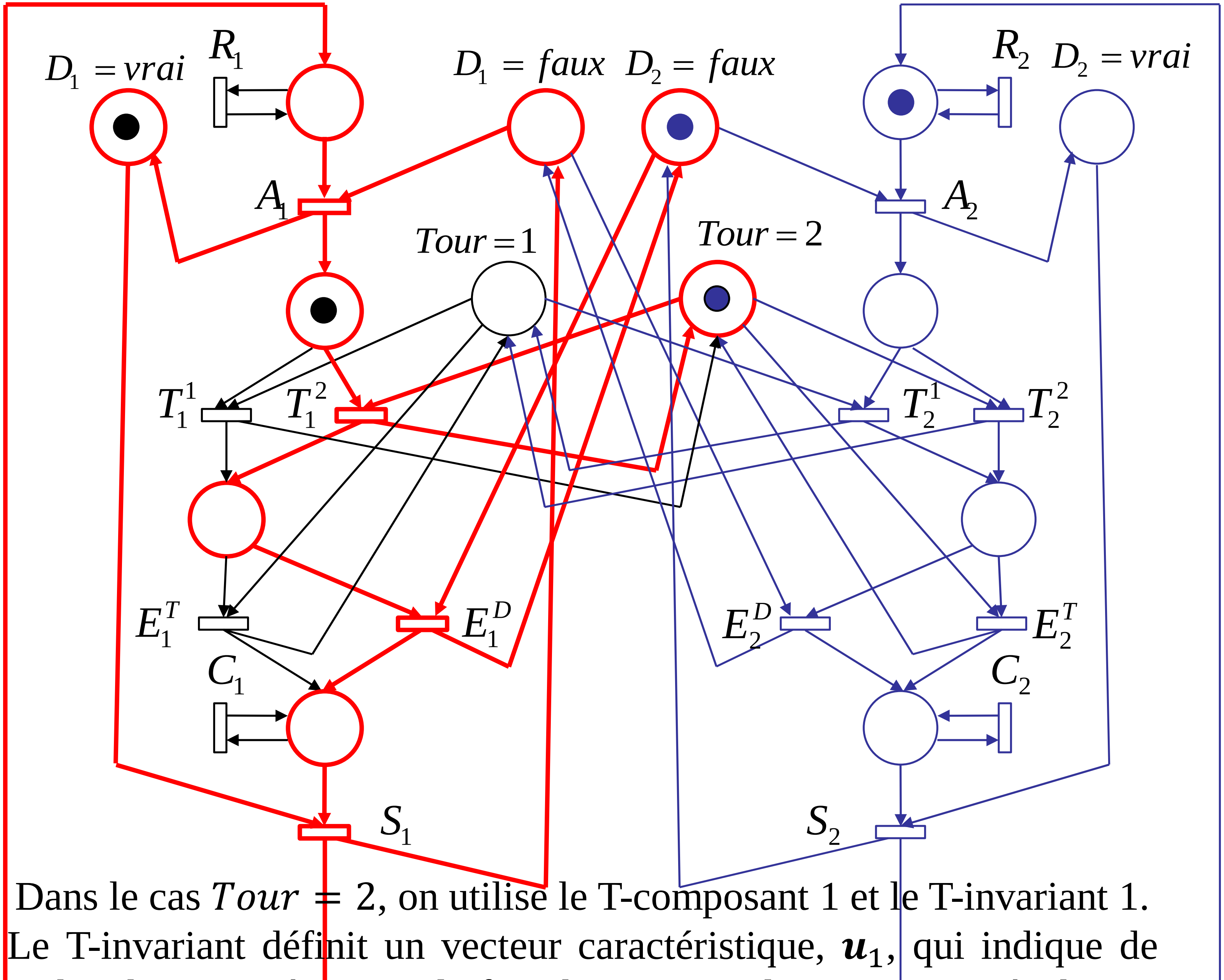

T-composant 1

J.-M. Delosme - SED

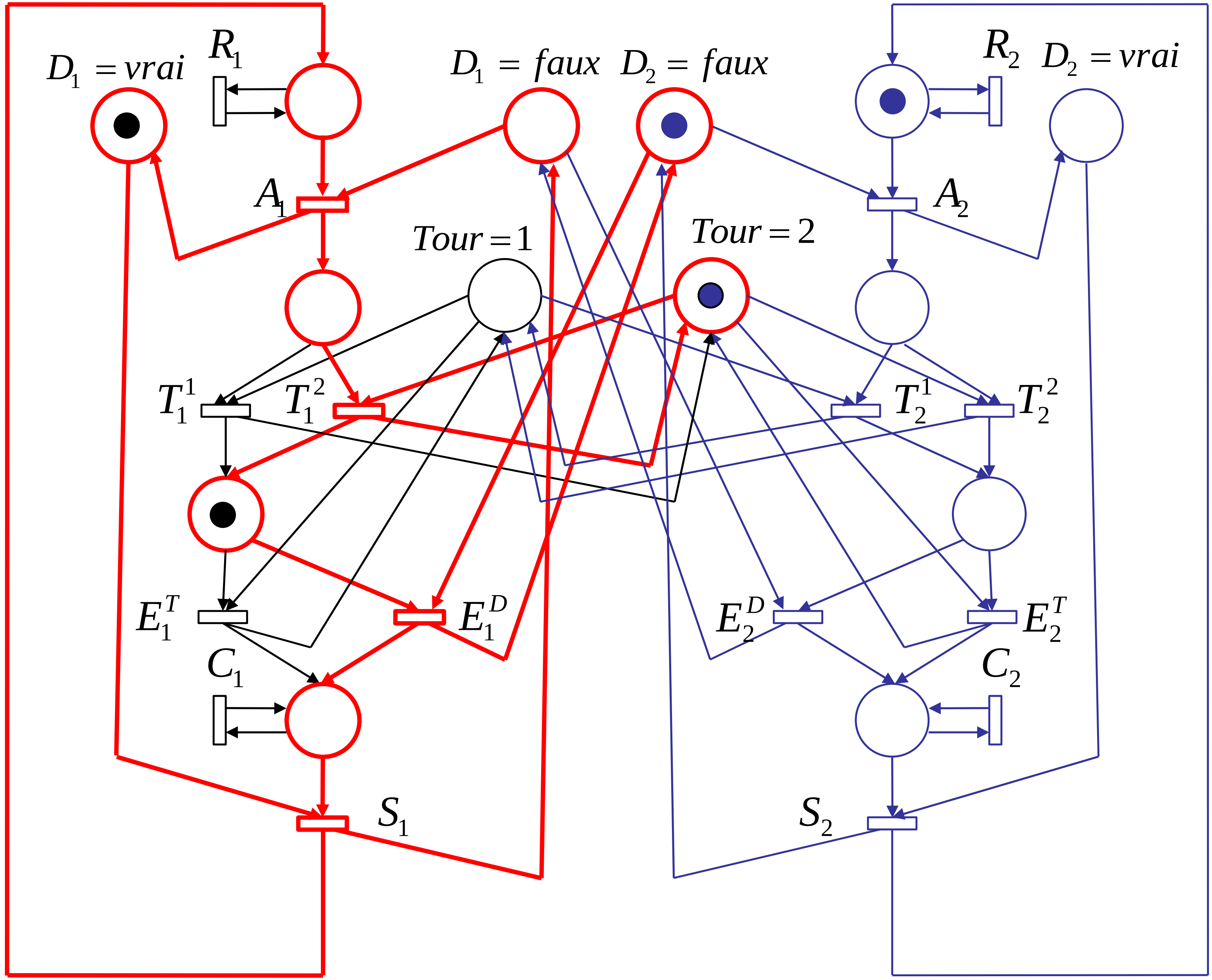



**Progression :** un processus en section restante ne peut empêcher l'autre processus de rentrer en section critique.

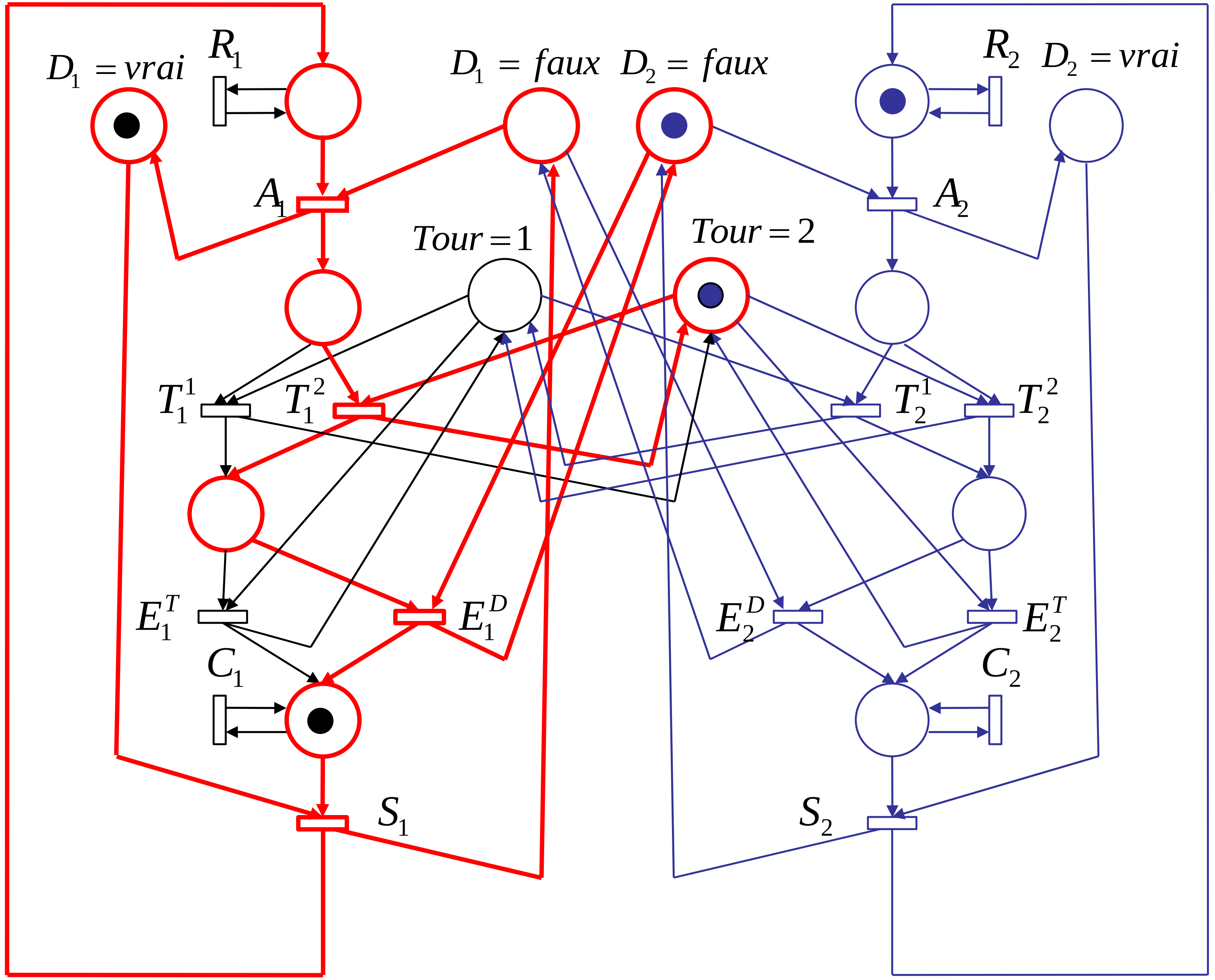
Lorsque $P_2$ est en section restante, les deux cas, $Tour = 1$ et $Tour = 2$, sont possibles. Si $Tour = 1$ nous avons vu que $P_1$ peut effectuer une itération et, à la fin, $Tour = 2$.



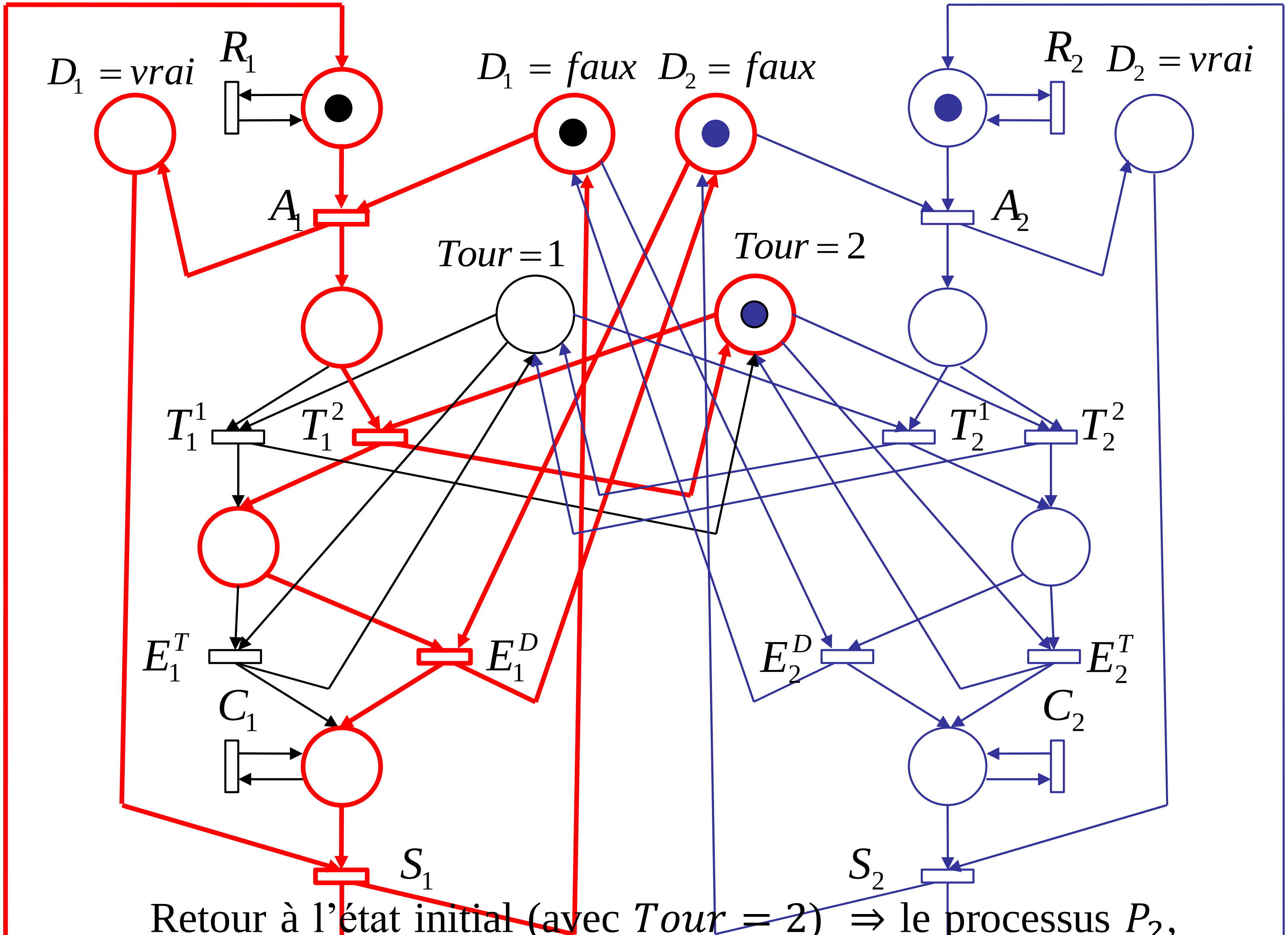

Dans le cas $Tour = 2$, on utilise le T-composant 1 et le T-invariant 1. Le T-invariant définit un vecteur caractéristique, $\boldsymbol{u}_1$, qui indique de rechercher une séquence de franchissements de transitions où chaque transition du T-composant est exécutée une fois.




J.-M. Delosme - SED




J.-M. Delosme - SED




Retour à l'état initial (avec $Tour = 2$) $\Rightarrow$ le processus $P_2$, en section restante, ne peut empêcher le processus $P_1$ de rentrer en section critique un nombre non-borné de fois.

La progression est donc assurée pour $P_1$.

$$
+\quad
\begin{matrix} A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2 \end{matrix}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

Recherche des autres T-invariants.

$$
\begin{matrix} A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2 \end{matrix}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

Détermination de T-invariant 2

$$
+\quad
\begin{matrix} A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2 \end{matrix}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

$$
\begin{array}{c}
A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2
\end{array}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

J.-M. Delosme - SED

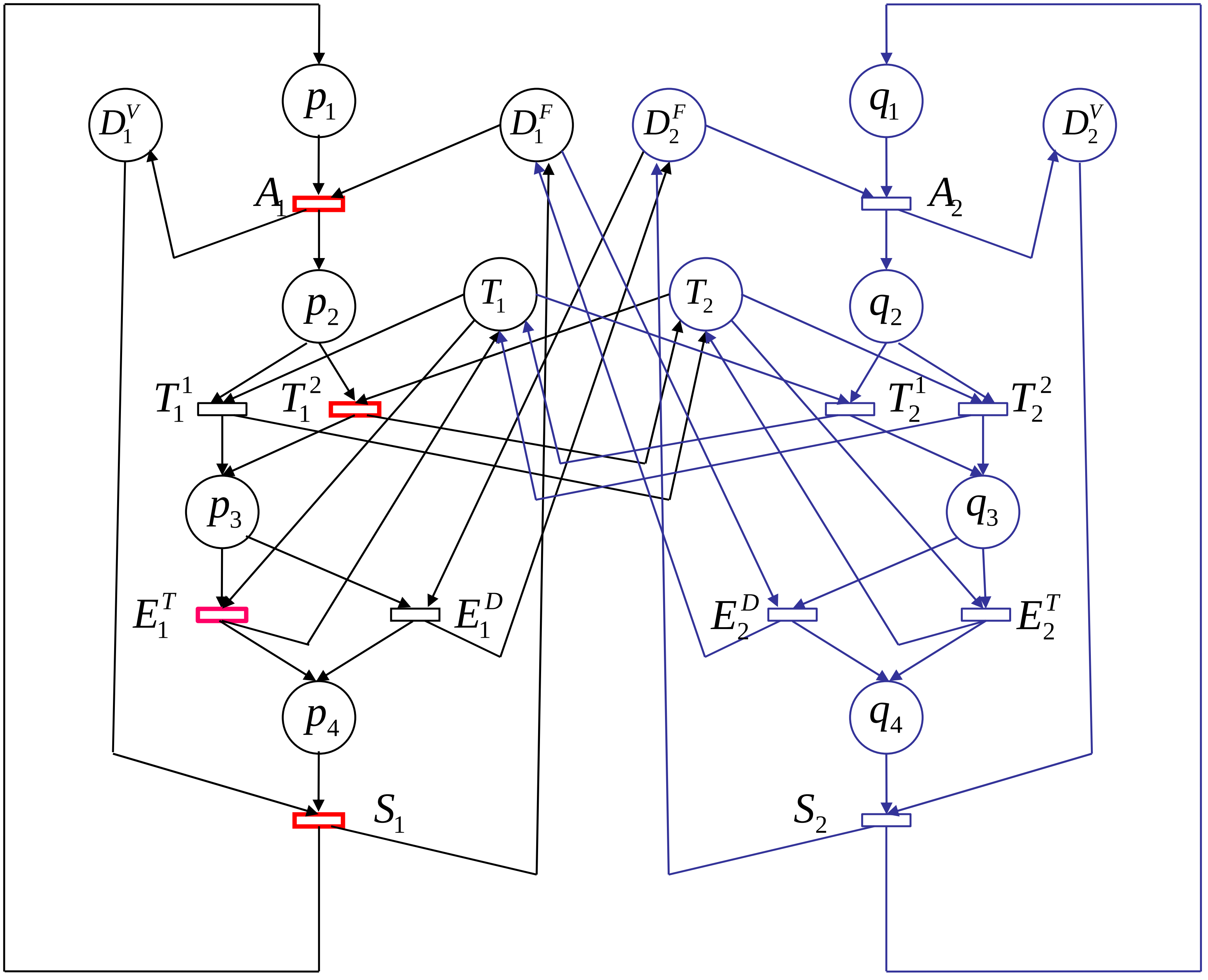
$$
+\quad
\begin{array}{c}
A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2
\end{array}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

J.-M. Delosme - SED

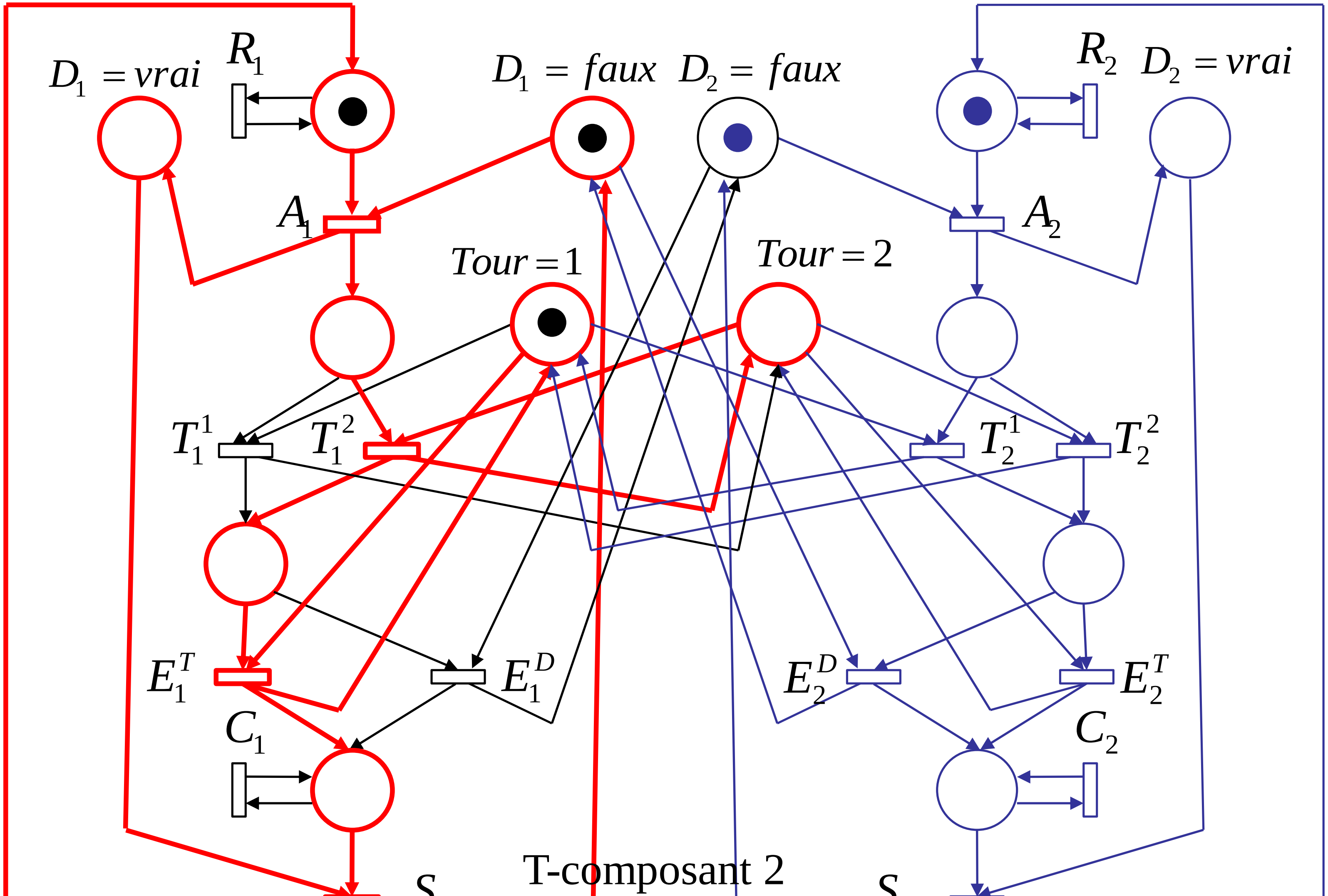
| | $D_1^F$ | $D_1^V$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $T_1$ | $T_2$ | $D_2^F$ | $D_2^V$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | −1 | 1 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^1$ | 0 | 0 | 0 | −1 | 1 | 0 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^2$ | 0 | 0 | 0 | −1 | 1 | 0 | 0 | **0** | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^T$ | 0 | 0 | 0 | 0 | −1 | 1 | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^D$ | 0 | 0 | 0 | 0 | −1 | 1 | 0 | 0 | **0** | 0 | 0 | 0 | 0 | 0 |
| $S_1$ | 1 | −1 | 1 | 0 | 0 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $A_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | −1 | 1 | 0 | 0 |
| $T_2^1$ | 0 | 0 | 0 | 0 | 0 | 0 | **0** | 0 | 0 | 0 | 0 | −1 | 1 | 0 |
| $T_2^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 0 | 0 | 0 | −1 | 1 | 0 |
| $E_2^T$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **0** | 0 | 0 | 0 | 0 | −1 | 1 |
| $E_2^D$ | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 |
| $S_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 1 | 0 | 0 | −1 |

T-invariant 2

$$\boldsymbol{u}_2 = [1,0,1,1,0,1,0,0,0,0,0,0]$$




J.-M. Delosme - SED




T-composant 2

Ce composant n'est pas utile : le P-invariant $\{T_1, T_2\}$ impose que, étant donné le marquage initial, il n'y aura toujours qu'un seul jeton dans l'ensemble des deux places. Il n'est donc pas possible de trouver une séquence de franchissements de transitions de vecteur caractéristique $\boldsymbol{u}_2$.

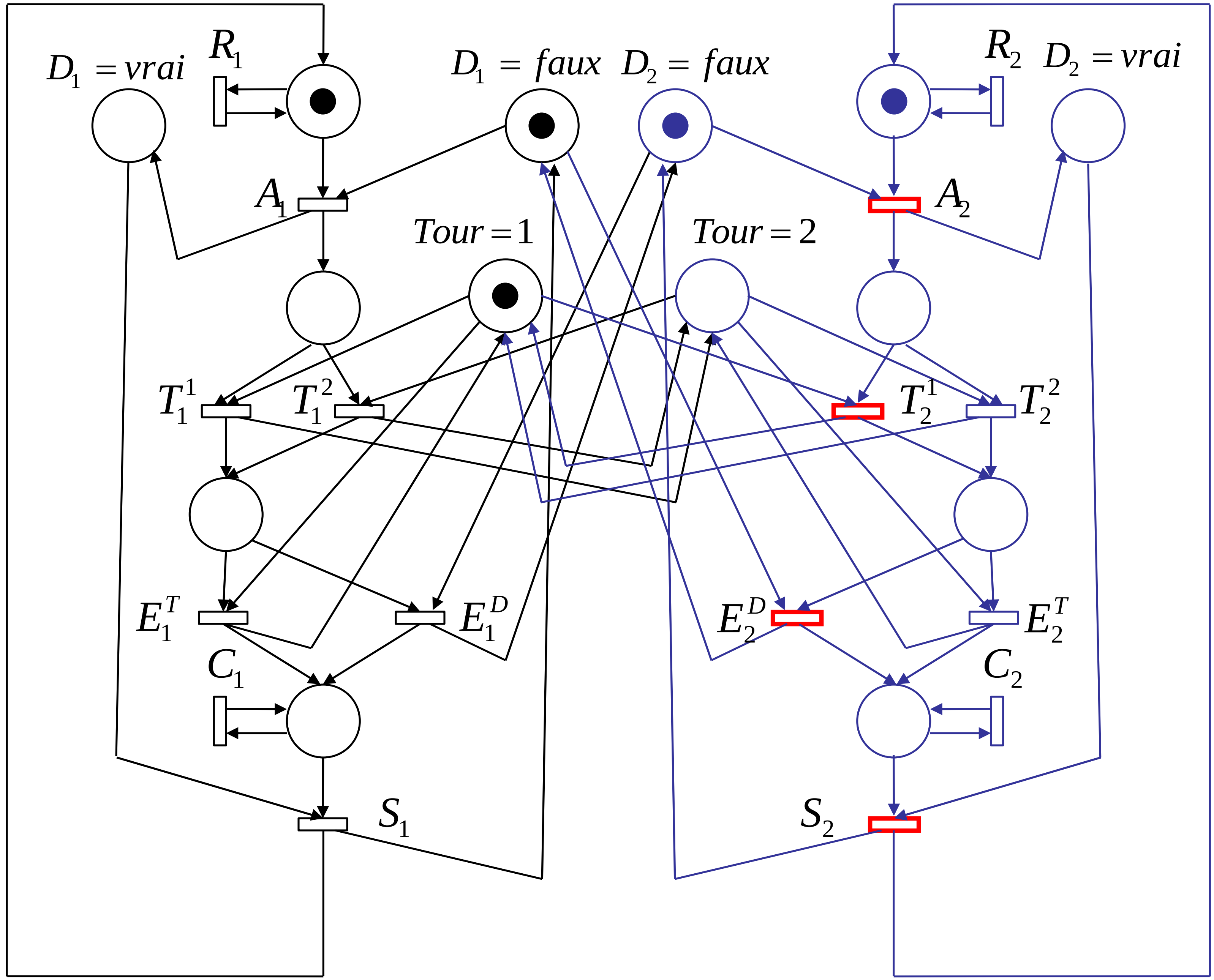
$$
+\quad
\begin{matrix} A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2 \end{matrix}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

Détermination du T-invariant 3

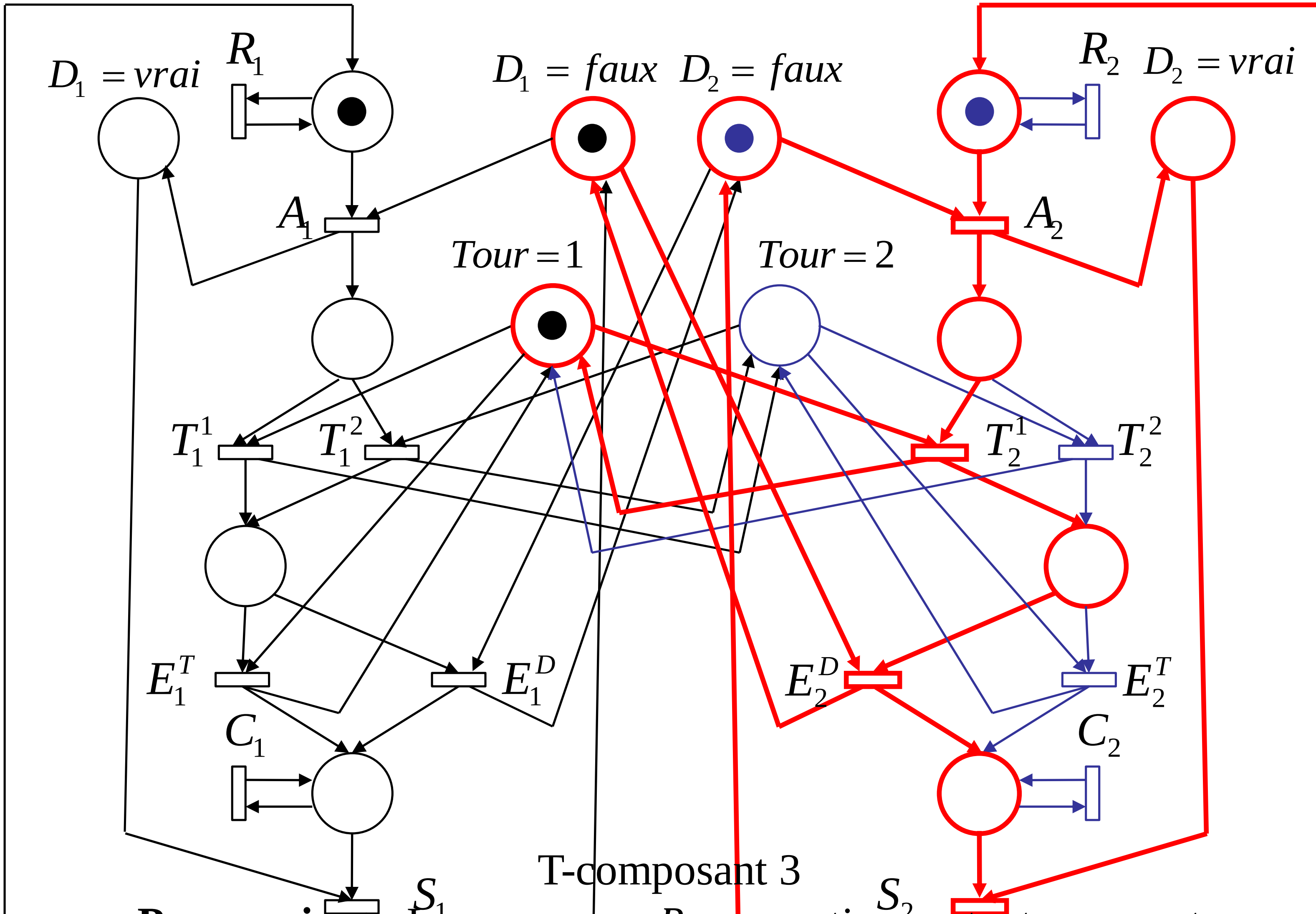
| | $D_1^F$ | $D_1^V$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $T_1$ | $T_2$ | $D_2^F$ | $D_2^V$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | –1 | 1 | –1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^1$ | 0 | 0 | 0 | –1 | 1 | 0 | –1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^2$ | 0 | 0 | 0 | –1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^T$ | 0 | 0 | 0 | 0 | –1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^D$ | 0 | 0 | 0 | 0 | –1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $S_1$ | 1 | –1 | 1 | 0 | 0 | –1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $A_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | –1 | 1 | –1 | 1 | 0 | 0 |
| $T_2^1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | –1 | 1 | 0 |
| $T_2^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | –1 | 0 | 0 | 0 | –1 | 1 | 0 |
| $E_2^T$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | –1 | 1 |
| $E_2^D$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | –1 | 1 |
| $S_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | –1 | 1 | 0 | 0 | –1 |

T-invariant 3

$$\boldsymbol{u}_3 = [0,0,0,0,0,0,1,1,0,0,1,1]$$



$D_1 = vrai$
$R_1$
$D_1 = faux$
$D_2 = faux$
$R_2$
$D_2 = vrai$
$A_1$
$A_2$
$Tour = 1$
$Tour = 2$
$T_1^1$
$T_1^2$
$T_2^1$
$T_2^2$
$E_1^T$
$E_1^D$
$E_2^D$
$E_2^T$
$C_1$
$C_2$
$S_1$
$S_2$




T-composant 3

**Progression :** Le processus $P_1$ en section restante ne peut empêcher le processus $P_2$ de rentrer en section critique.

La preuve, similaire à celle où $P_2$ demeure en section restante, utilise le T-composant 3 et le T-invariant 3.

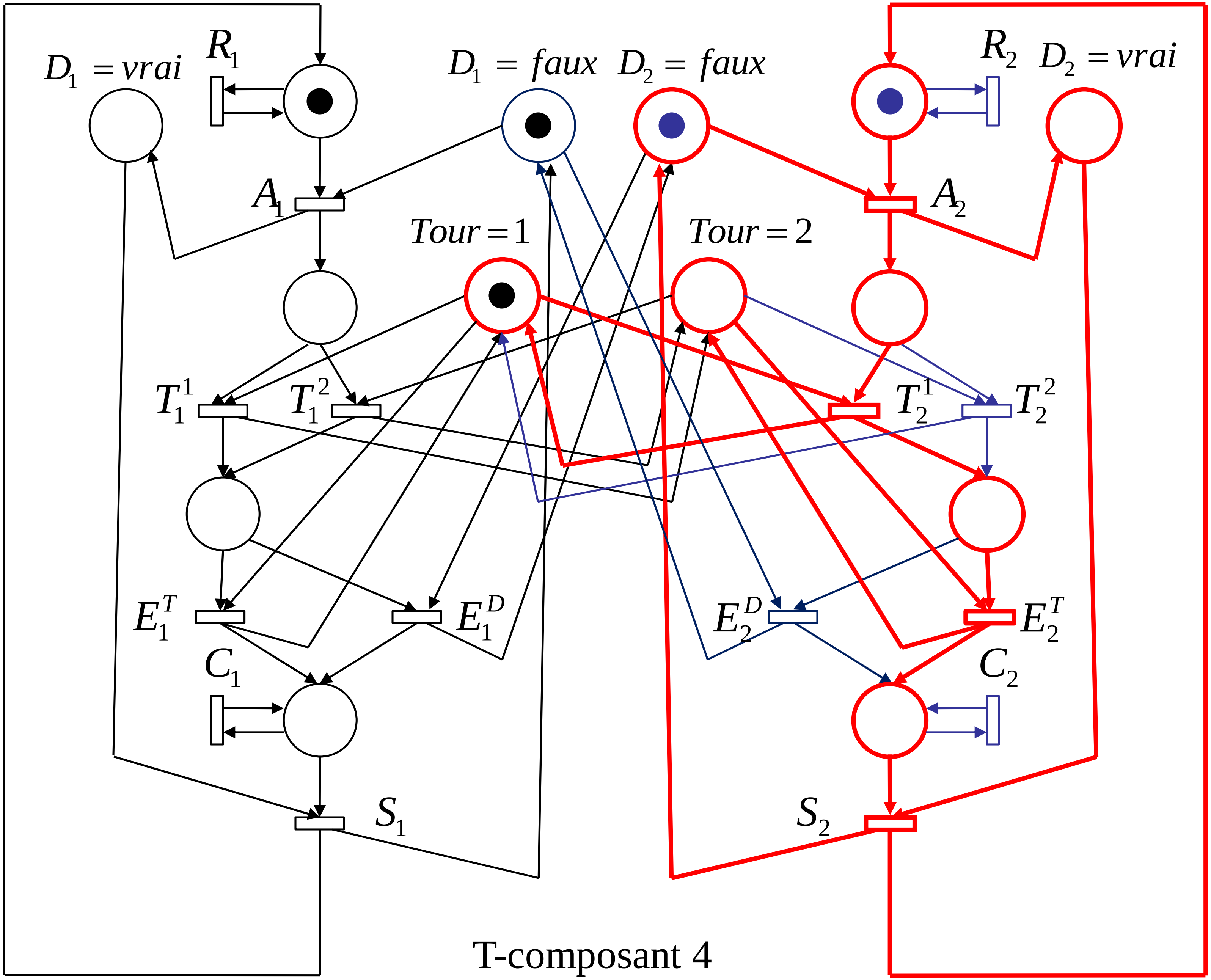
$$
+\quad
\begin{matrix} A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2 \end{matrix}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

Détermination du T-invariant 4

| | $D_1^F$ | $D_1^V$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $T_1$ | $T_2$ | $D_2^F$ | $D_2^V$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | −1 | 1 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^1$ | 0 | 0 | 0 | −1 | 1 | 0 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^2$ | 0 | 0 | 0 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^T$ | 0 | 0 | 0 | 0 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^D$ | 0 | 0 | 0 | 0 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $S_1$ | 1 | −1 | 1 | 0 | 0 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $A_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | −1 | 1 | 0 | 0 |
| $T_2^1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | 0 |
| $T_2^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 0 | 0 | 0 | −1 | 1 | 0 |
| $E_2^T$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 |
| $E_2^D$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 |
| $S_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 1 | 0 | 0 | −1 |

T-invariant 4

$$\boldsymbol{u}_4 = [0,0,0,0,0,0,1,1,0,1,0,1]$$




T-composant 4
Ne nous est pas utile.

J.-M. Delosme - SED

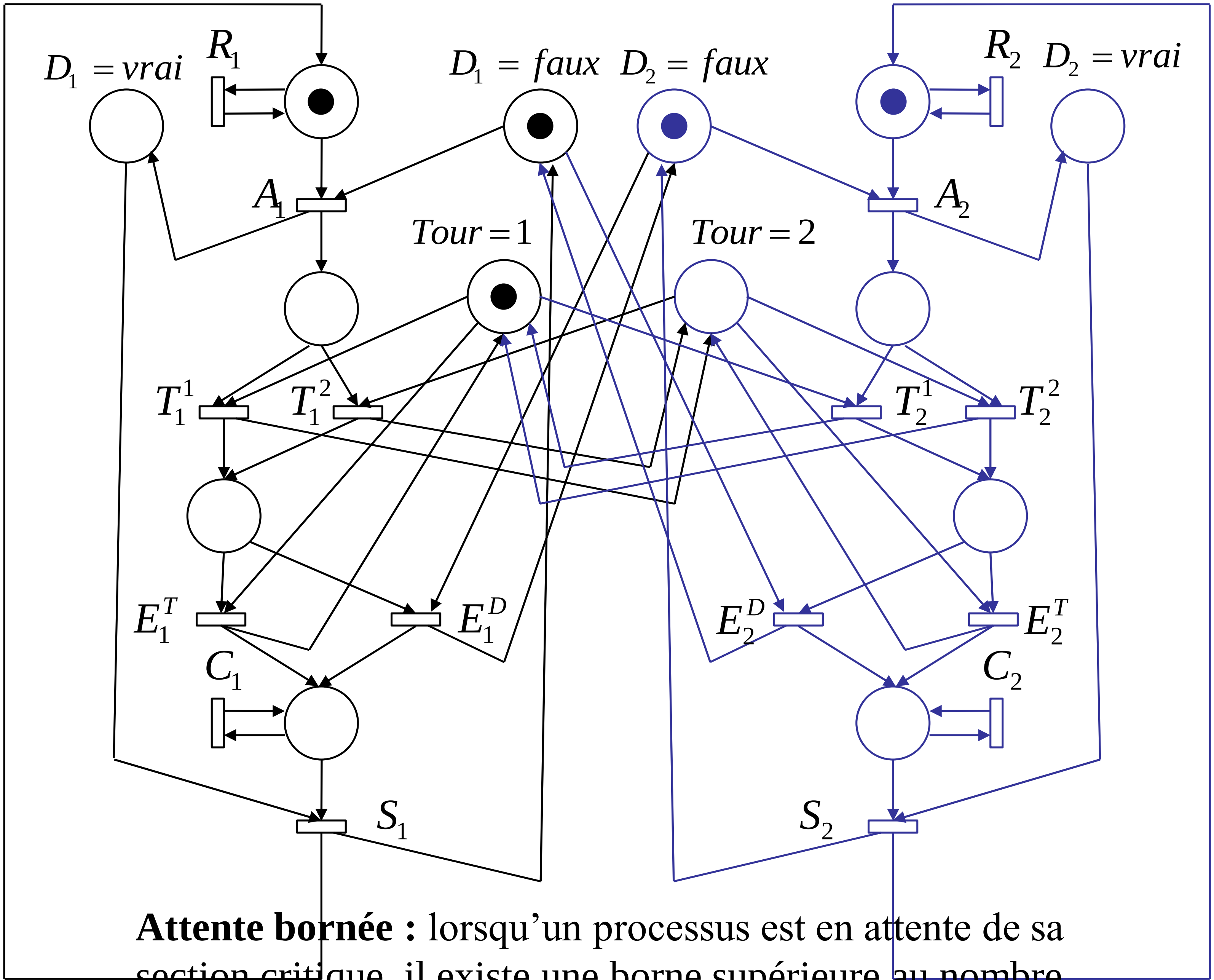
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $A_1$ | $-1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^1$ | 0 | $-1$ | 0 | 0 | 0 | 0 | 0 |
| $T_1^2$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $E_1^T$ | 0 | 0 | $-1$ | 1 | 0 | 0 | 0 |
| $E_1^D$ | 0 | 0 | $-1$ | 1 | 0 | 0 | 0 |
| $S_1$ | 1 | 0 | 0 | $-1$ | 0 | 0 | 0 |
| $A_2$ | 0 | 0 | 0 | 0 | $-1$ | 0 | 0 |
| $T_2^1$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $T_2^2$ | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| $E_2^T$ | 0 | 0 | 0 | 0 | 0 | $-1$ | 1 |
| $E_2^D$ | 0 | 0 | 0 | 0 | 0 | $-1$ | 1 |
| $S_2$ | 0 | 0 | 0 | 0 | 1 | 0 | $-1$ |

Détermination du T-invariant 5

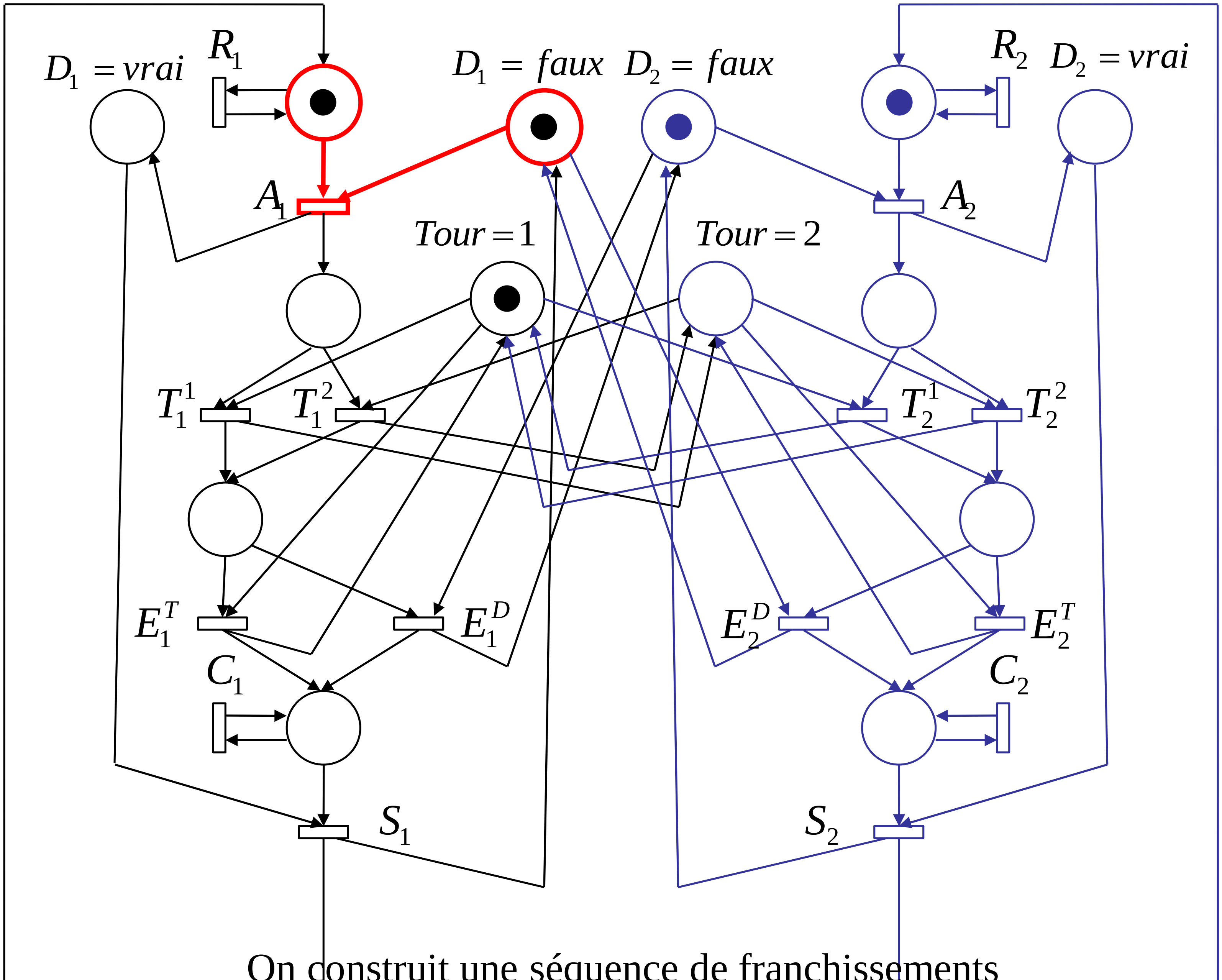
| | $D_1^F$ | $D_1^V$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $T_1$ | $T_2$ | $D_2^F$ | $D_2^V$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | $-1$ | 1 | $-1$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^1$ | 0 | 0 | 0 | $-1$ | 1 | 0 | $-1$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_1^2$ | 0 | 0 | 0 | $-1$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^T$ | 0 | 0 | 0 | 0 | $-1$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1^D$ | 0 | 0 | 0 | 0 | $-1$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $S_1$ | 1 | $-1$ | 1 | 0 | 0 | $-1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $A_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $-1$ | 1 | $-1$ | 1 | 0 | 0 |
| $T_2^1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $-1$ | 1 | 0 |
| $T_2^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | $-1$ | 0 | 0 | 0 | $-1$ | 1 | 0 |
| $E_2^T$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $-1$ | 1 |
| $E_2^D$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $-1$ | 1 |
| $S_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | $-1$ | 1 | 0 | 0 | $-1$ |

T-invariant 5

$$\boldsymbol{u}_5 = [1,1,0,0,1,1,1,0,1,0,1,1]$$

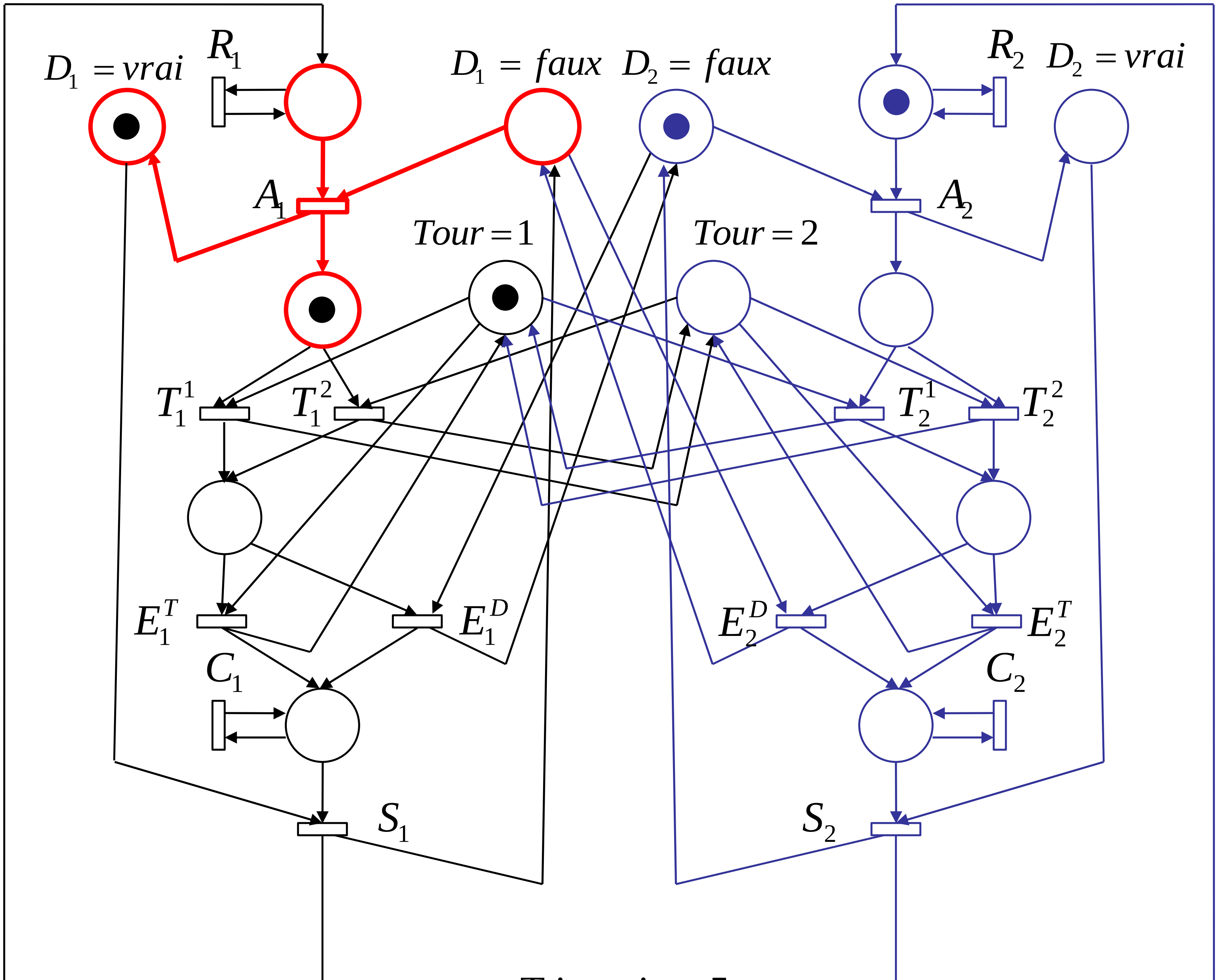



**Attente bornée :** lorsqu'un processus est en attente de sa section critique, il existe une borne supérieure au nombre de fois où l'autre processus exécute sa section critique.

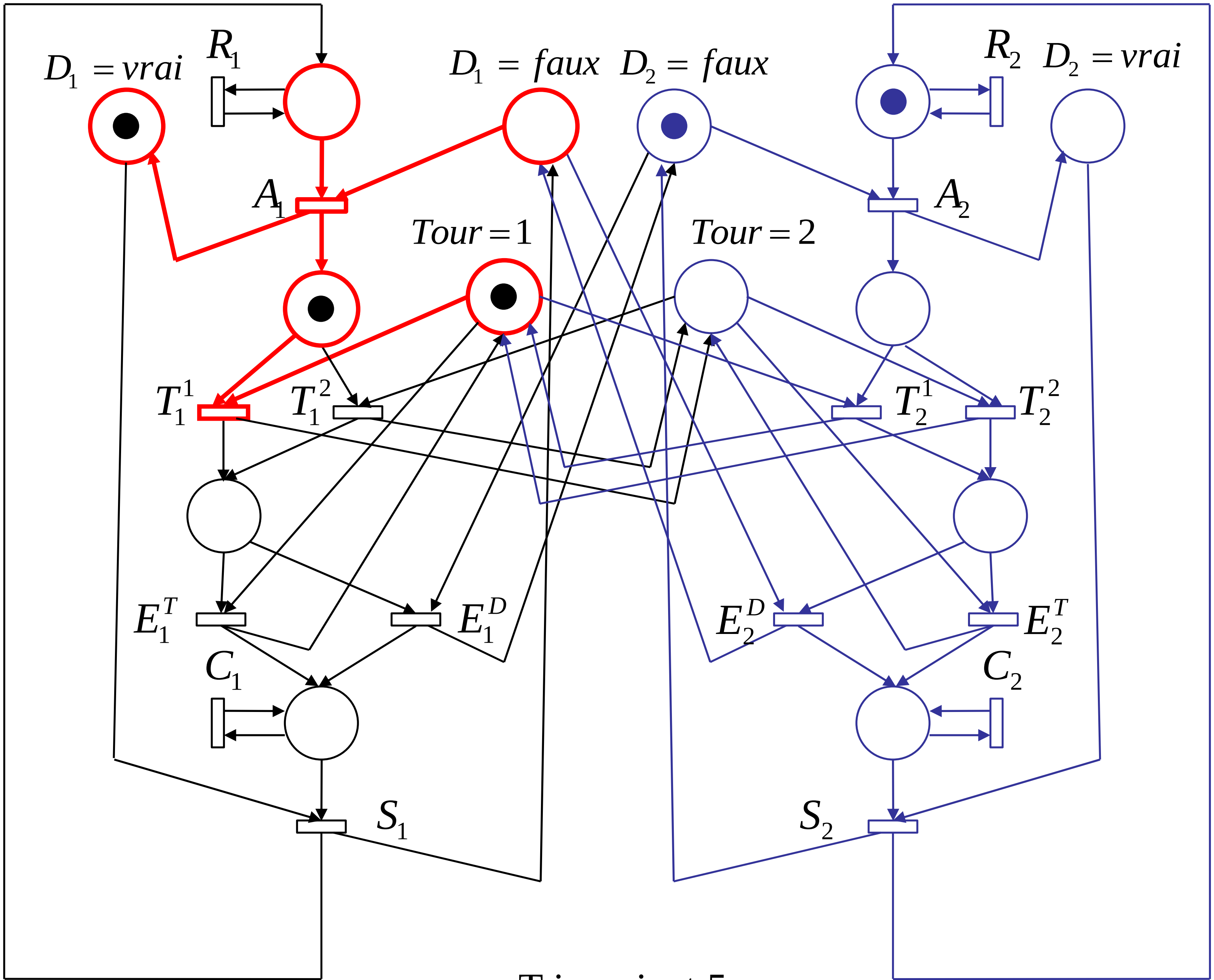



On construit une séquence de franchissements
de transitions de vecteur caractéristique $\boldsymbol{u}_5$.

J.-M. Delosme - SED

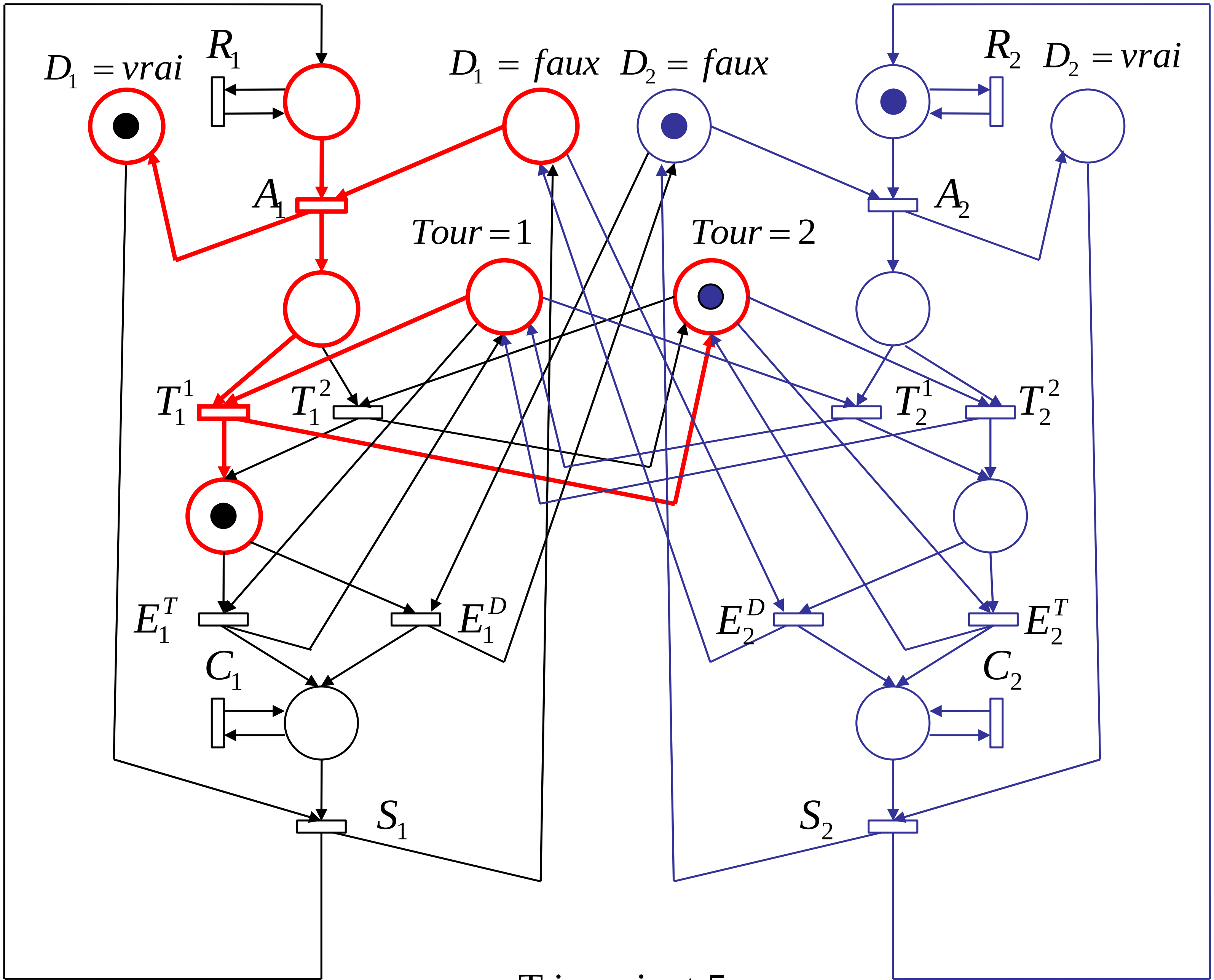



T-invariant 5



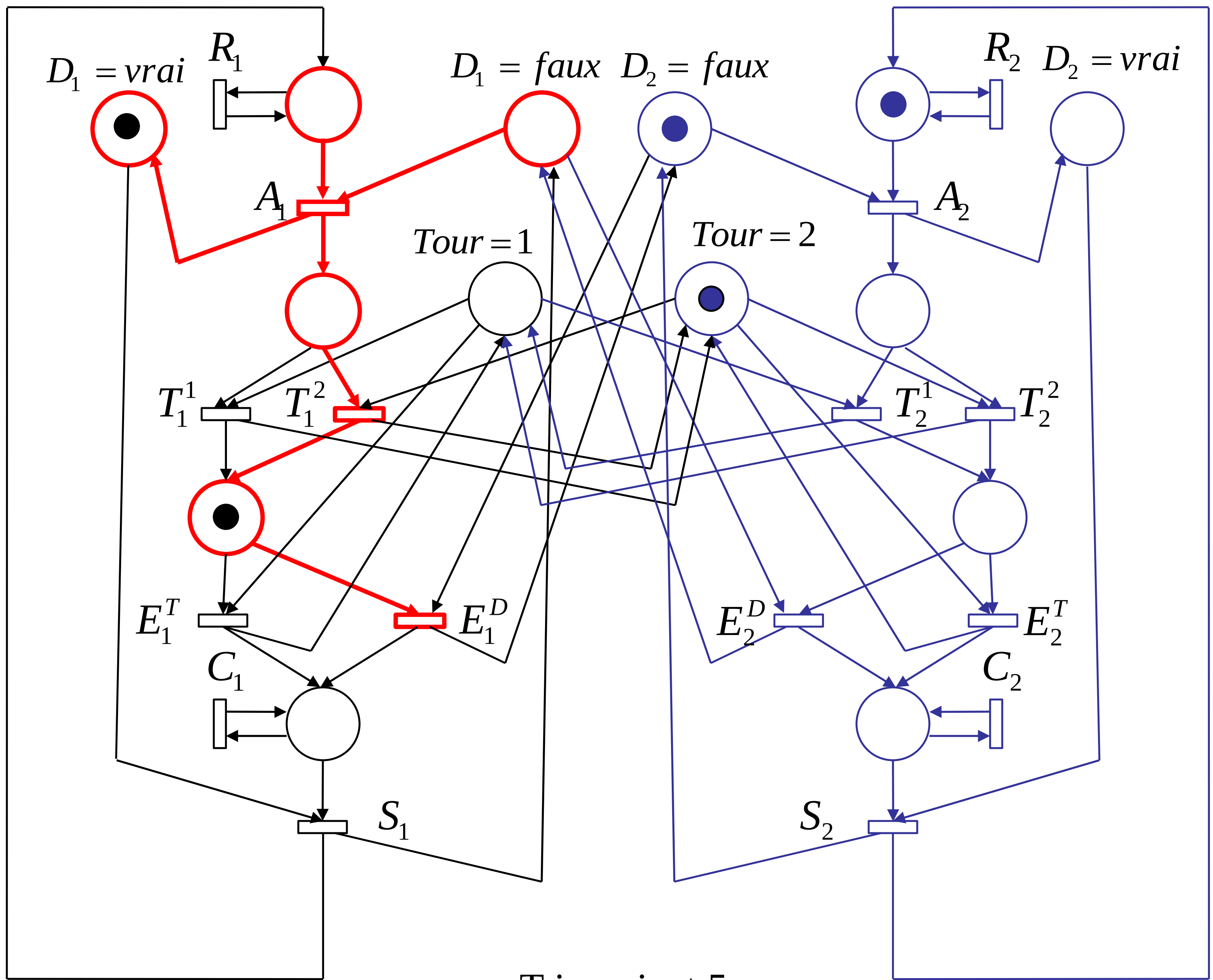

T-invariant 5

J.-M. Delosme - SED

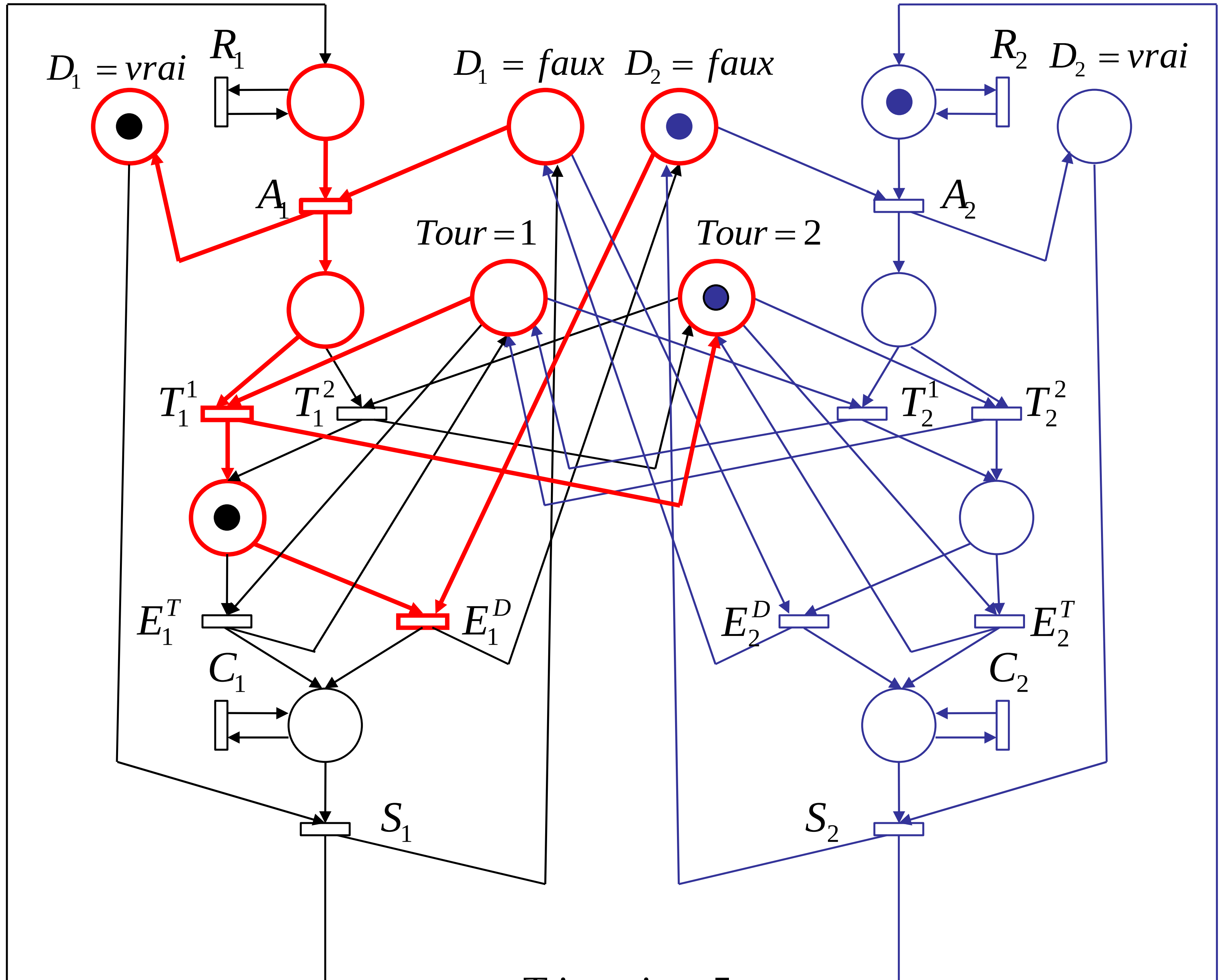



T-invariant 5

J.-M. Delosme - SED

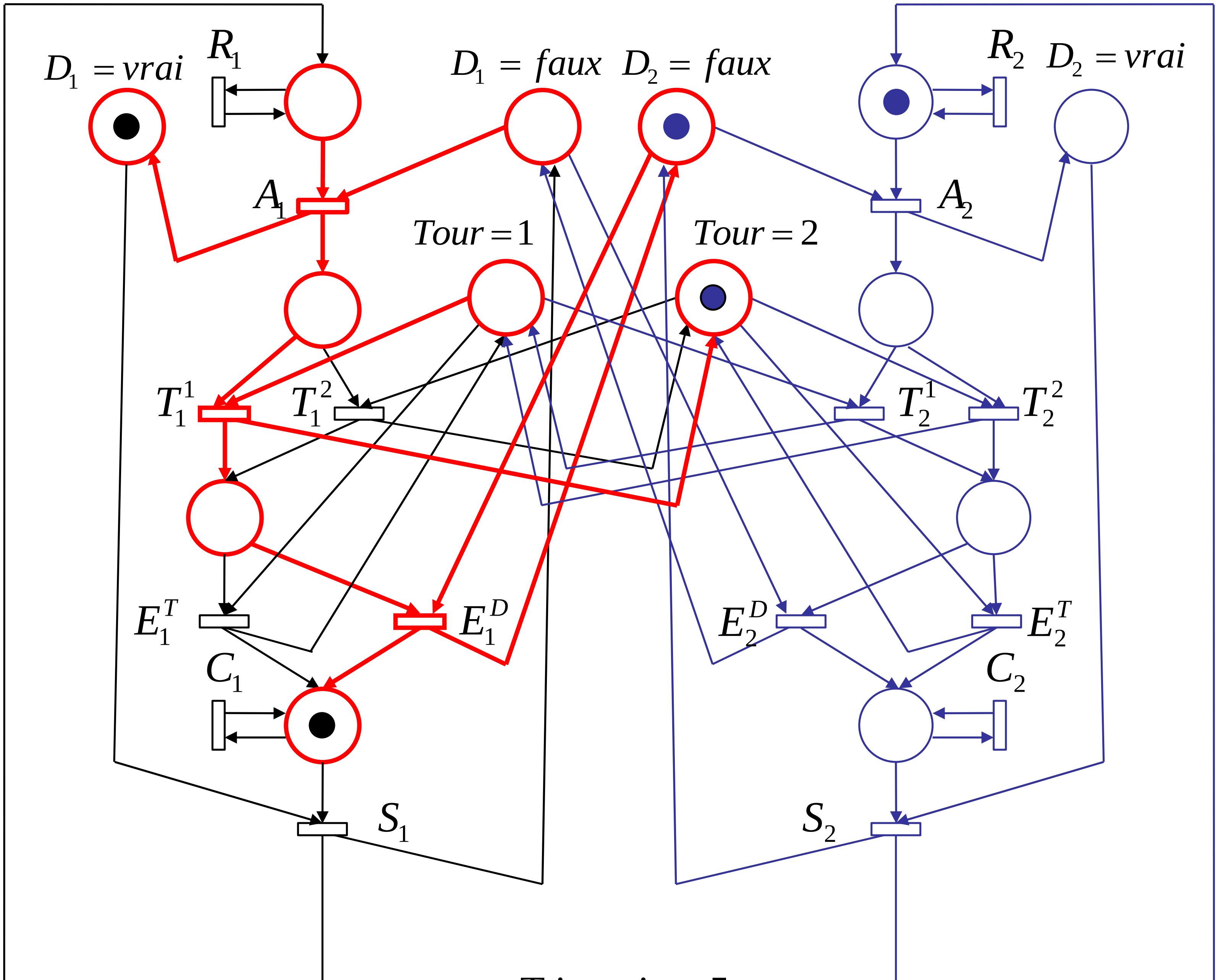



T-invariant 5



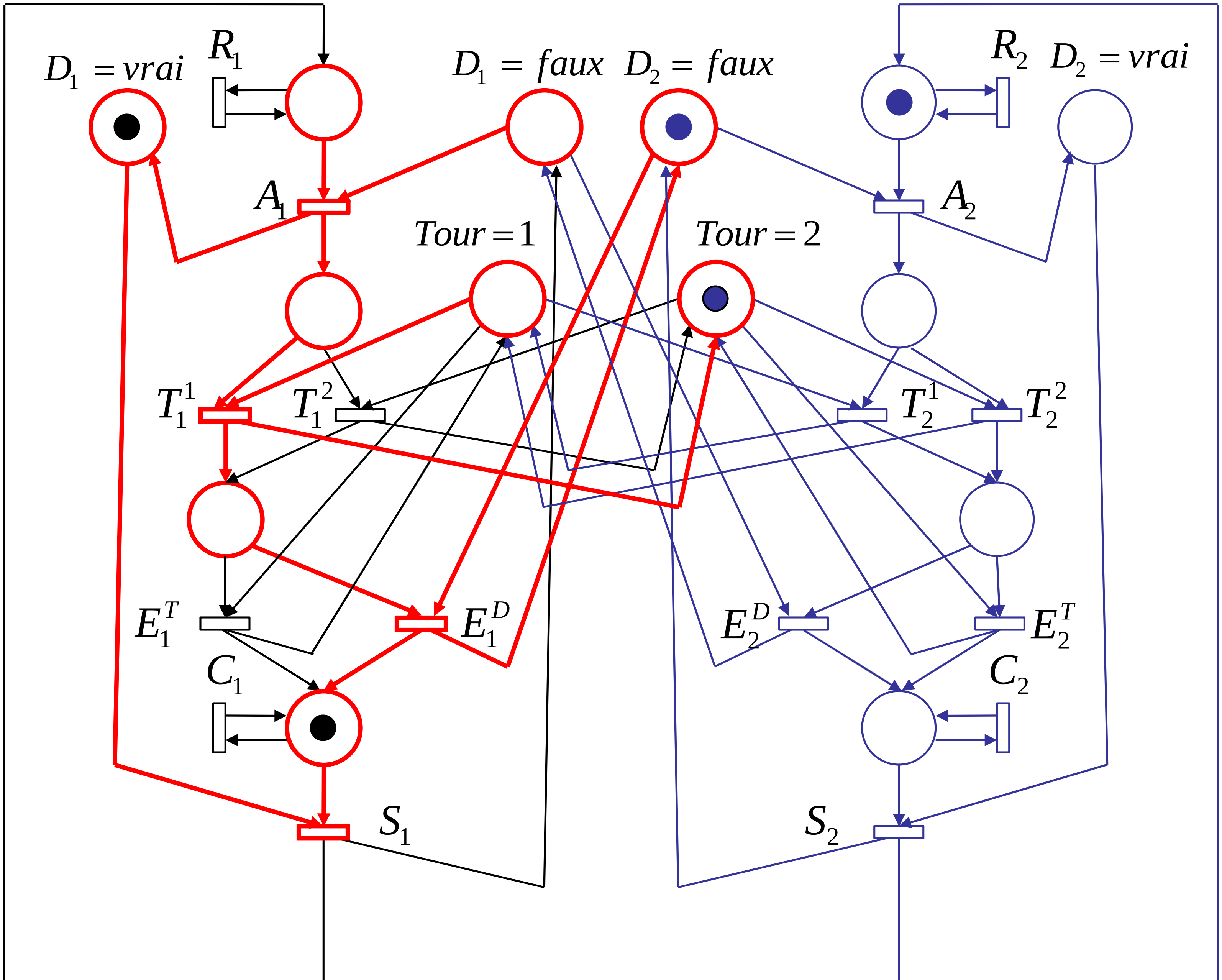

T-invariant 5

J.-M. Delosme - SED

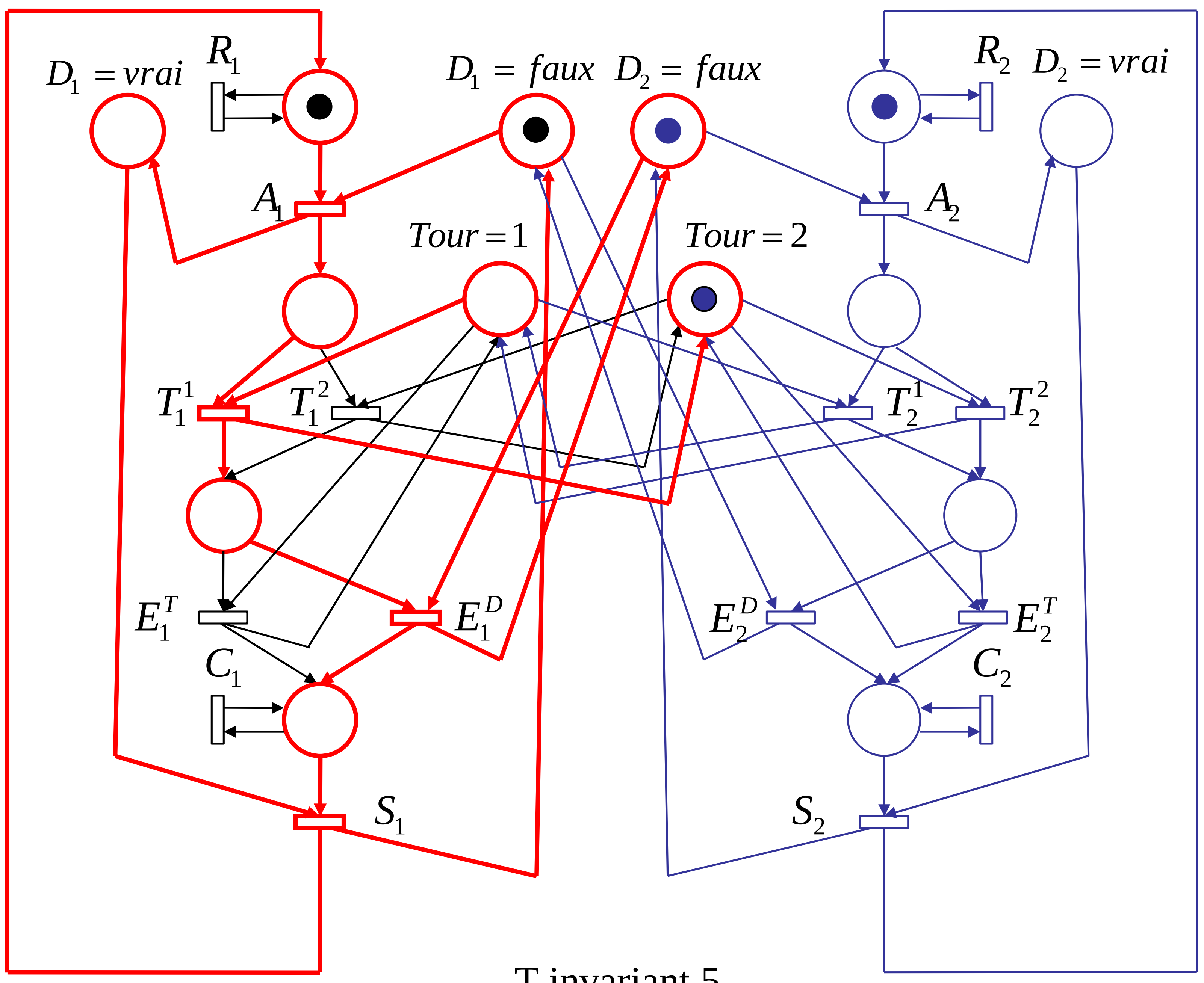



T-invariant 5

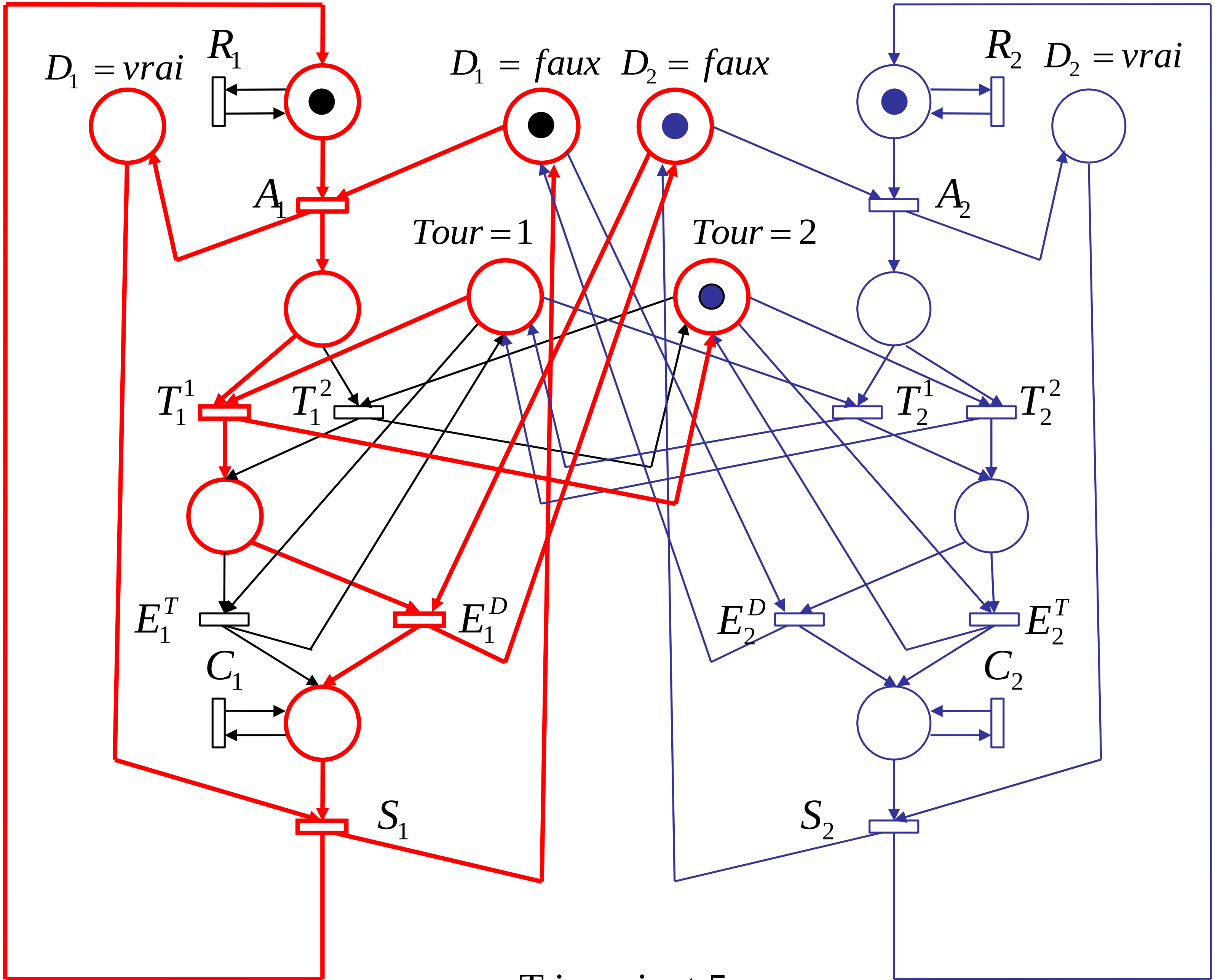
J.-M. Delosme - SED



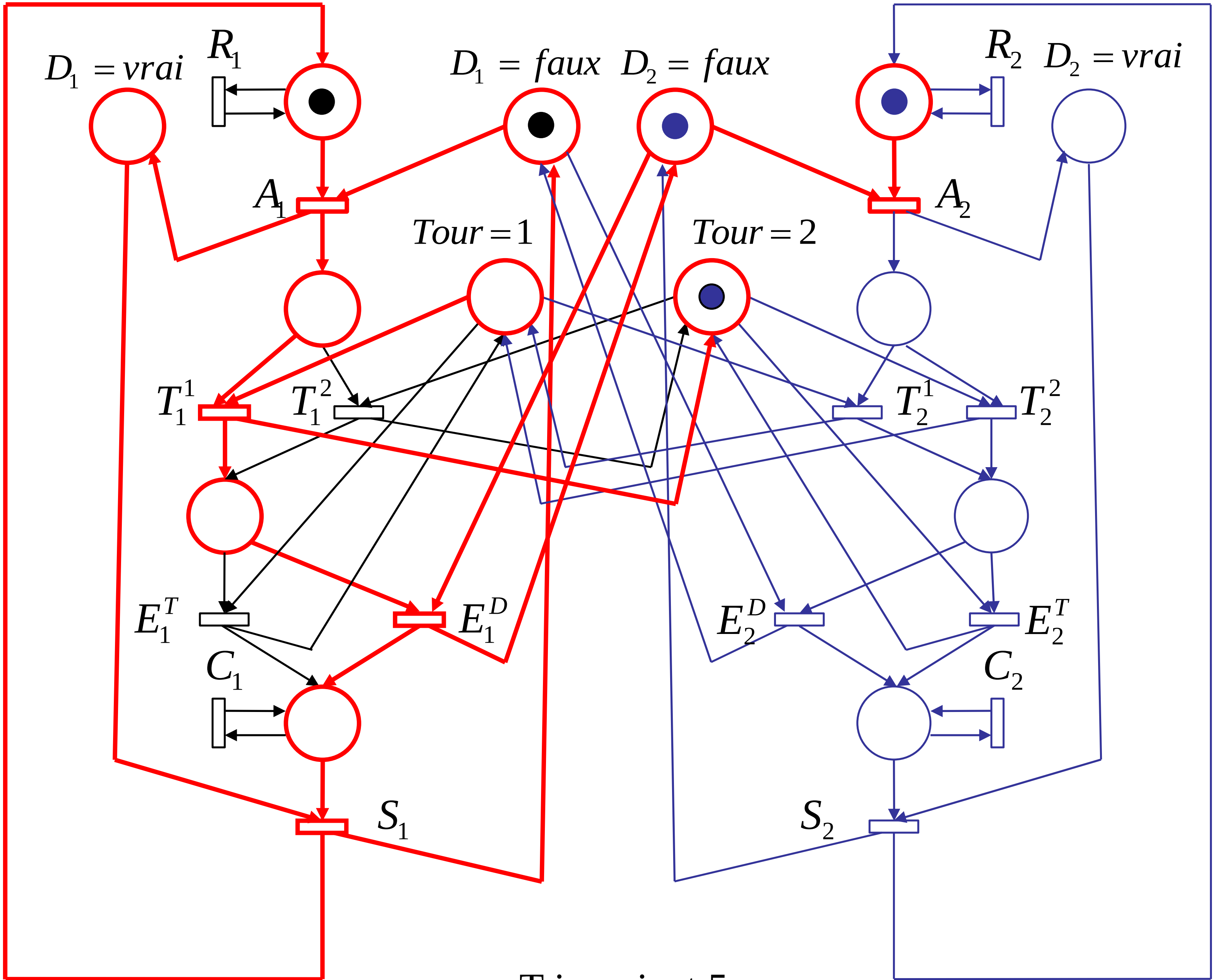

T-invariant 5



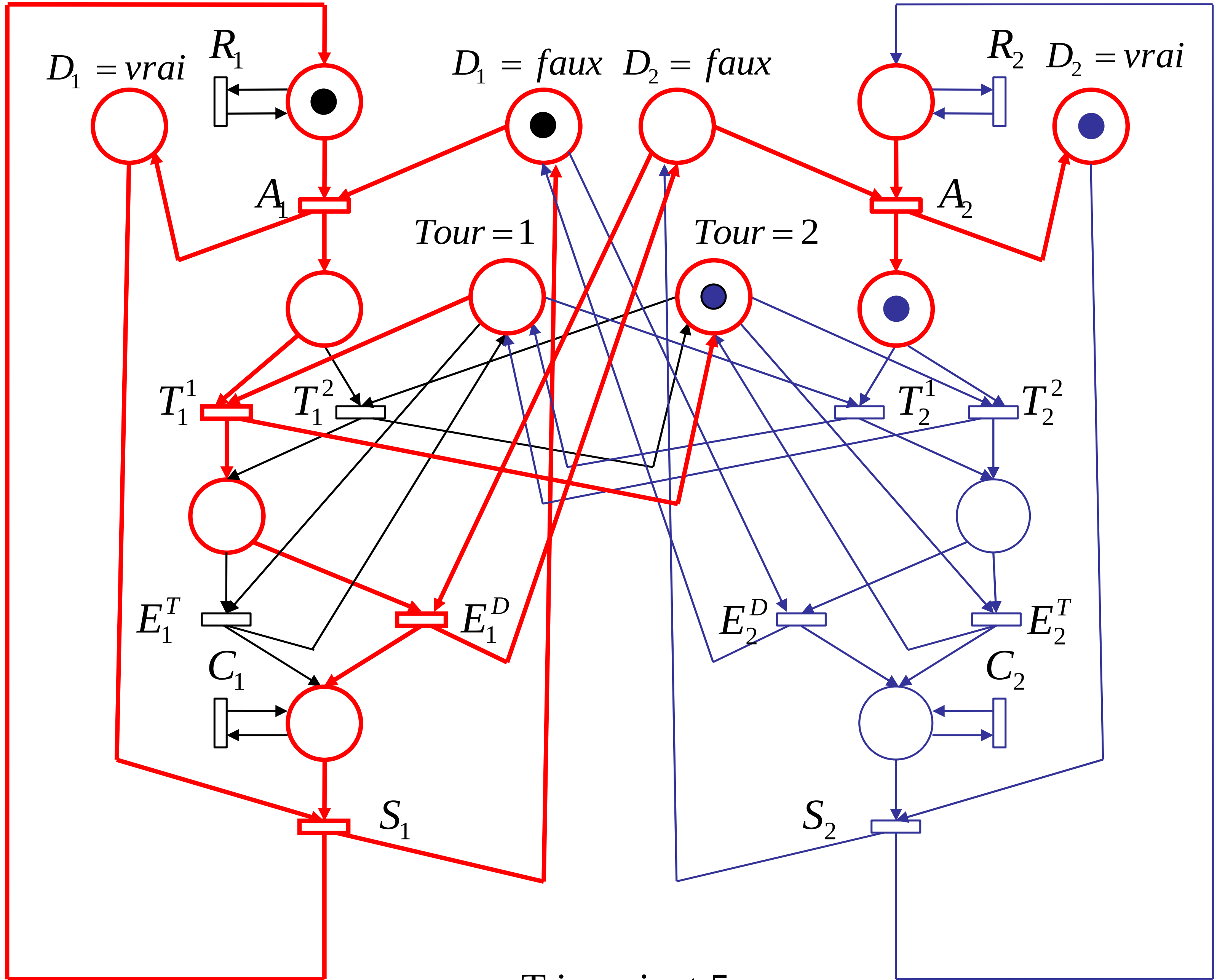

T-invariant 5



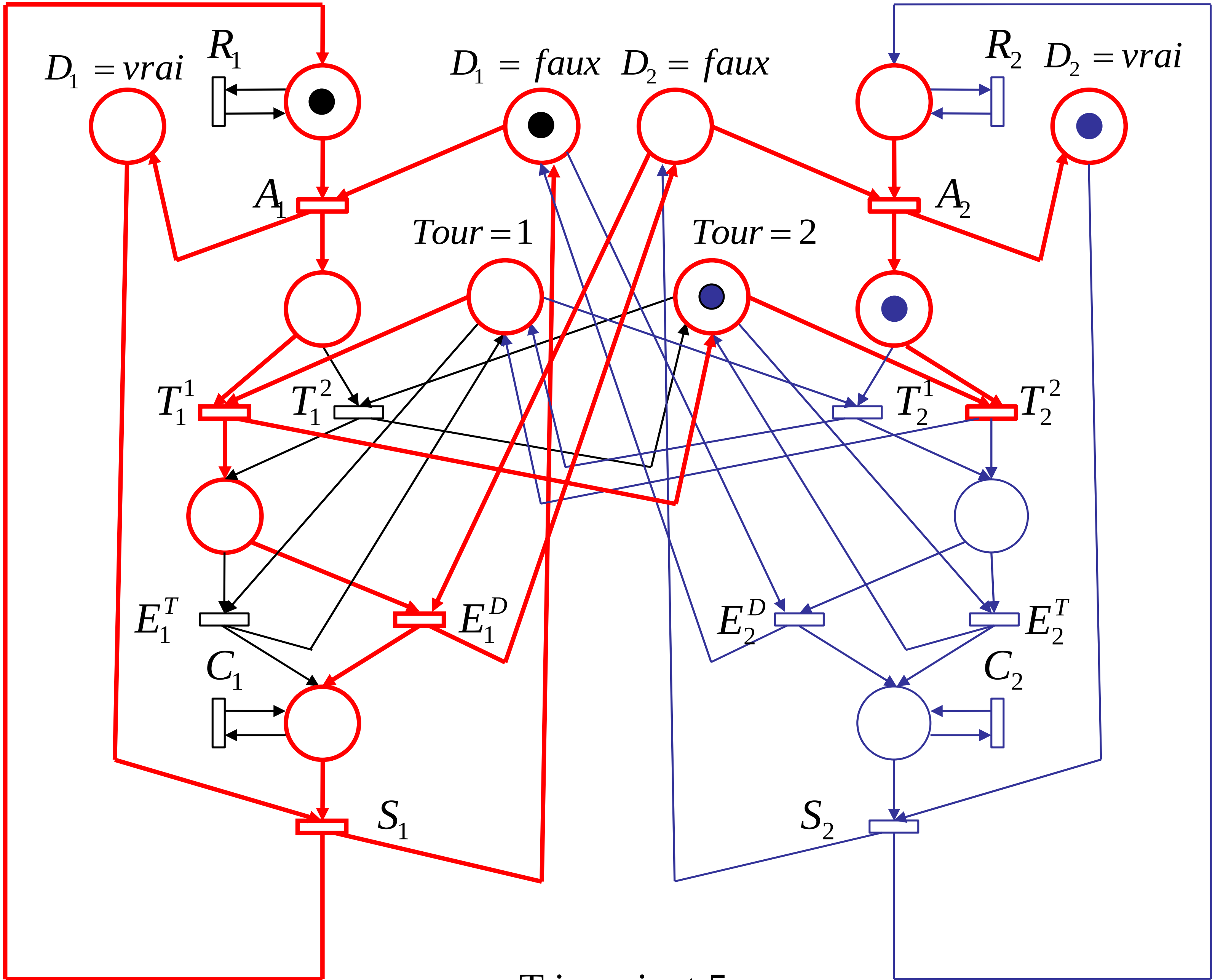

T-invariant 5




T-invariant 5

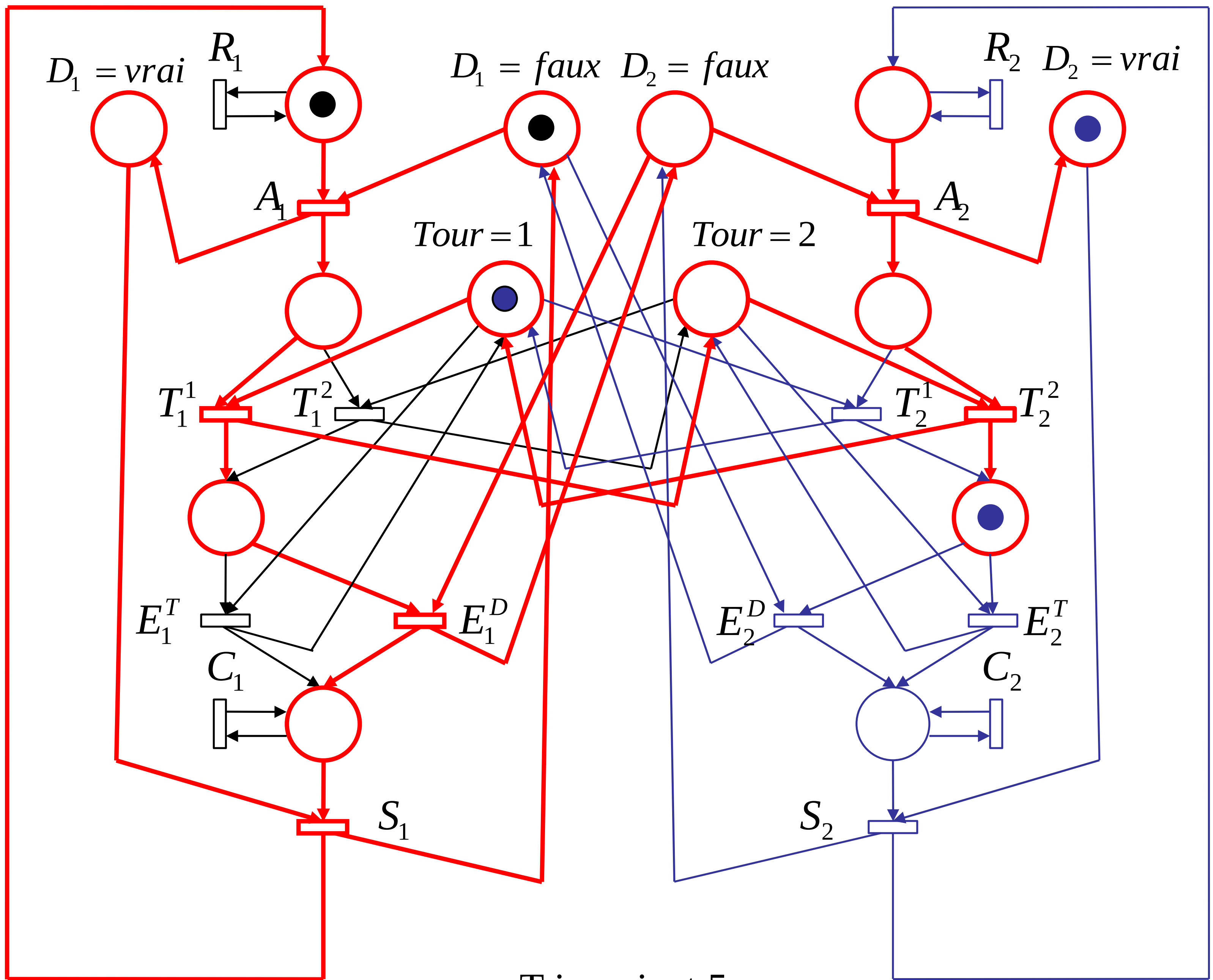



T-invariant 5



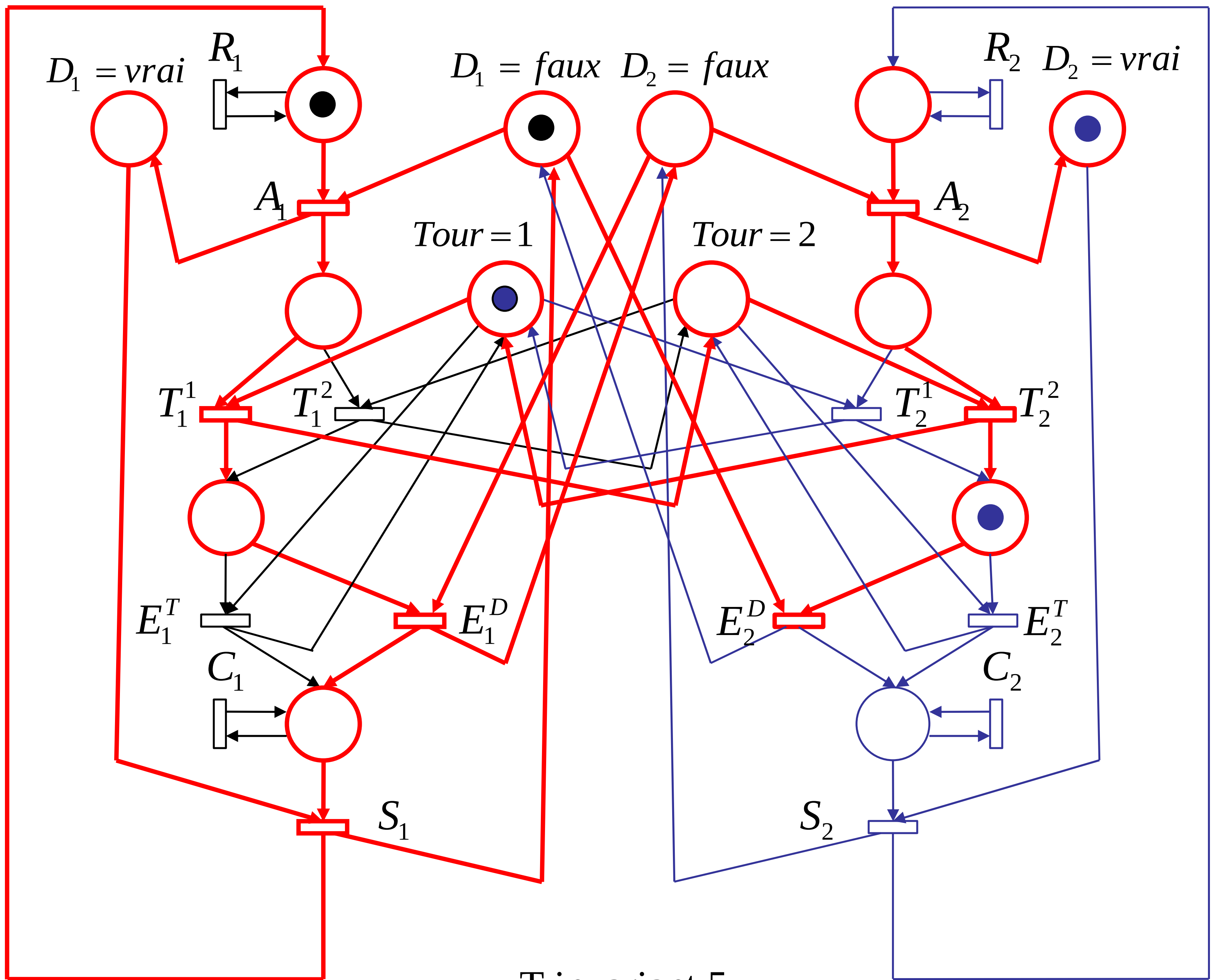

T-invariant 5

J.-M. Delosme - SED



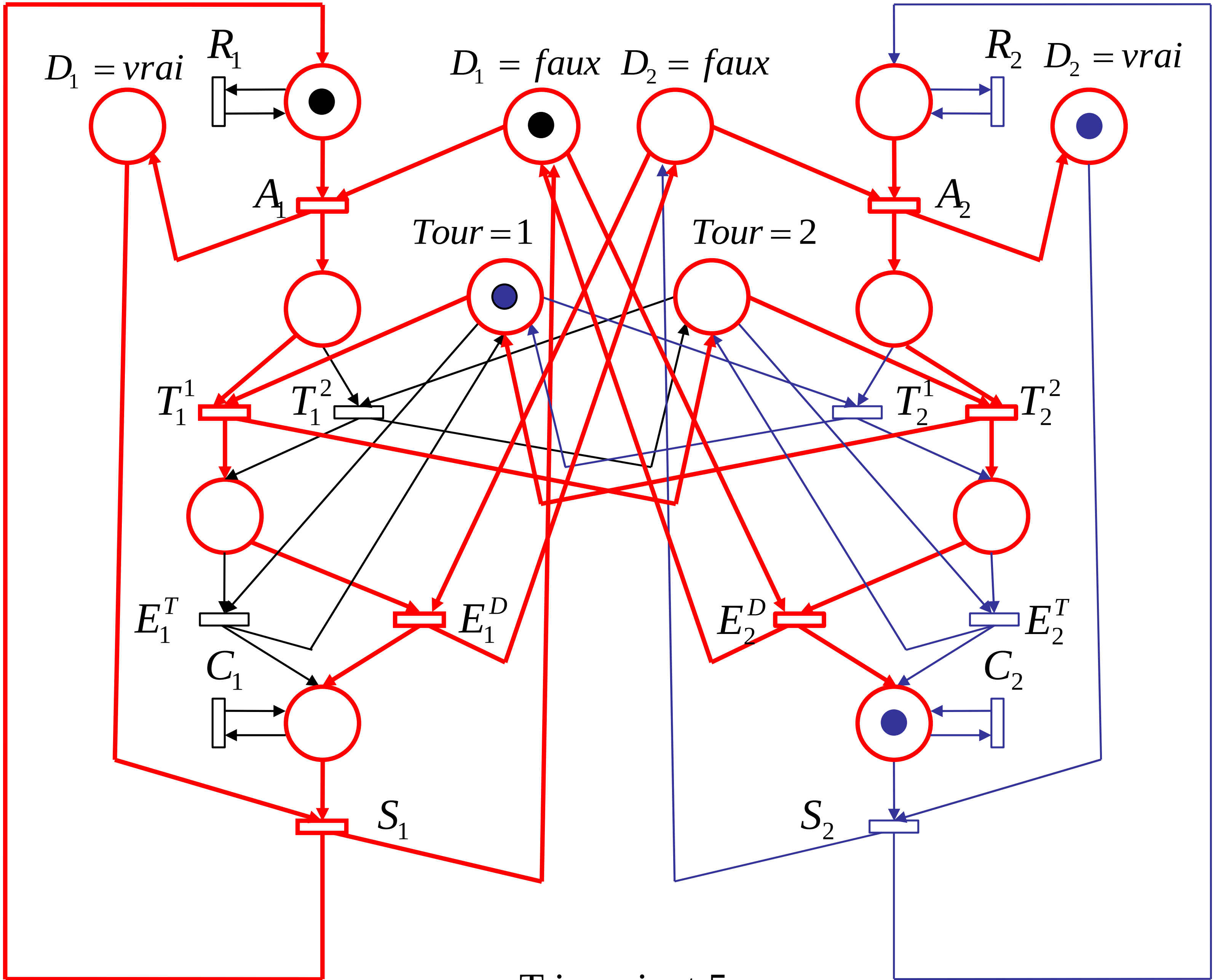

T-invariant 5

J.-M. Delosme - SED



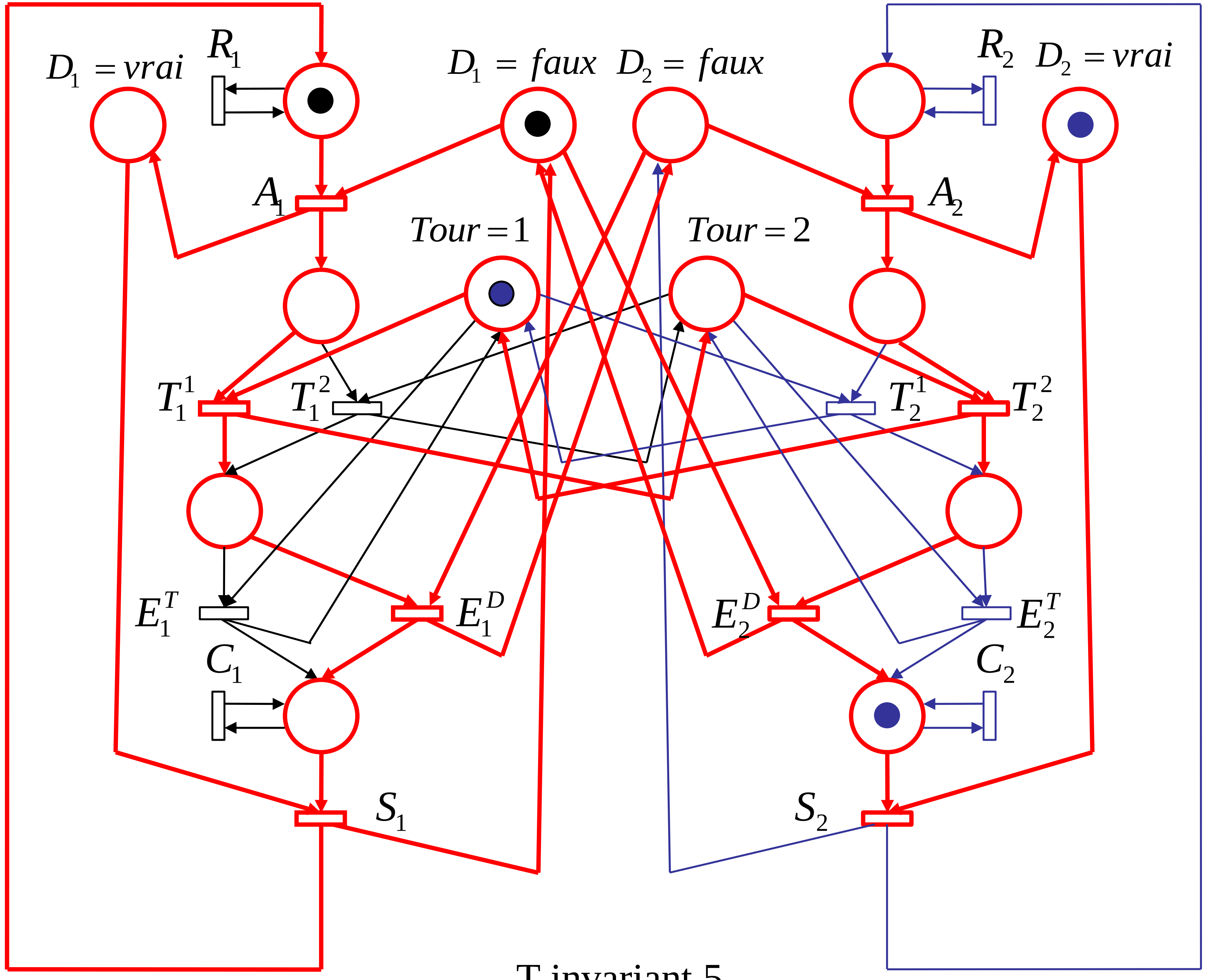

T-invariant 5

(processus est en attente de sa section critique)

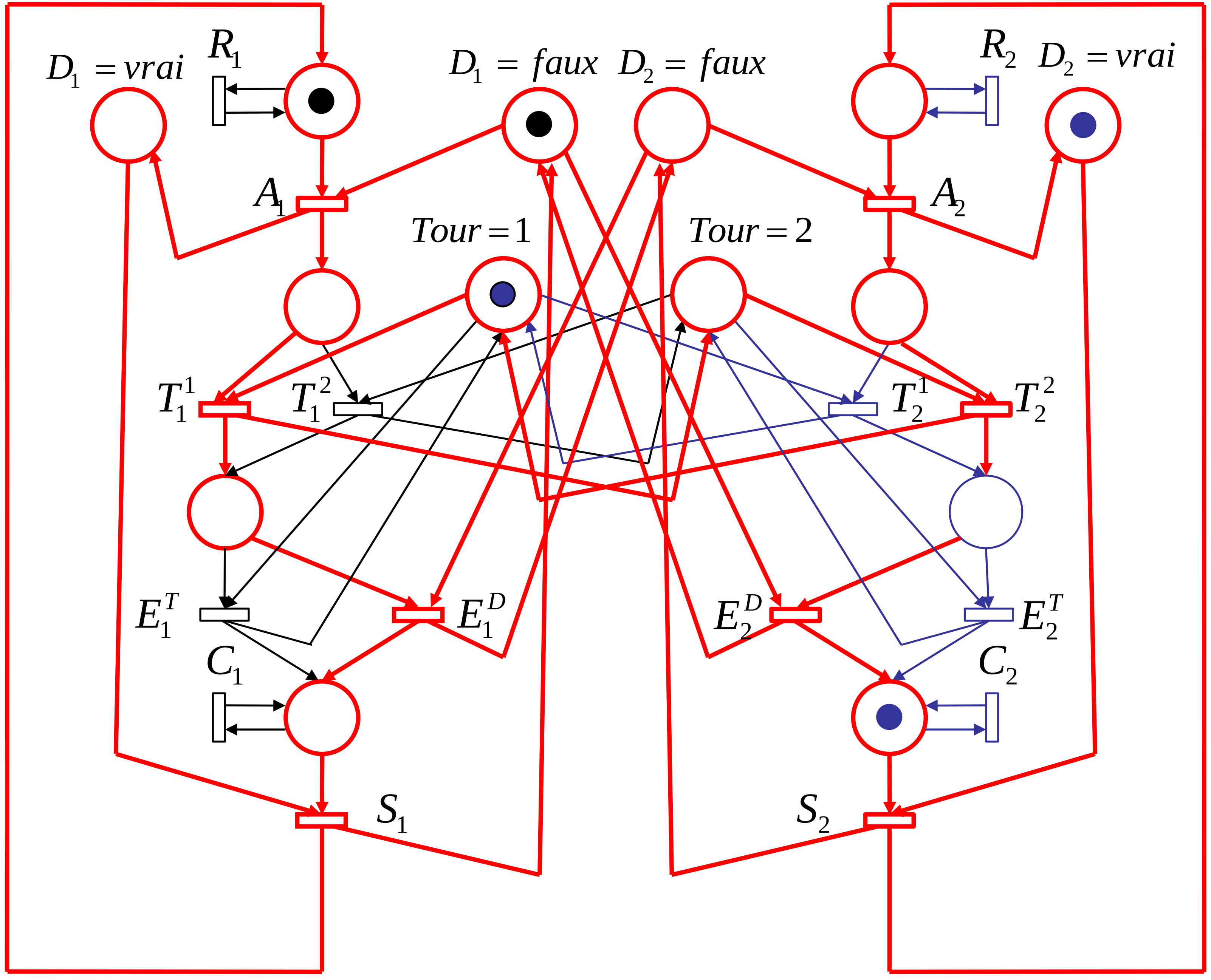



T-invariant 5

J.-M. Delosme - SED

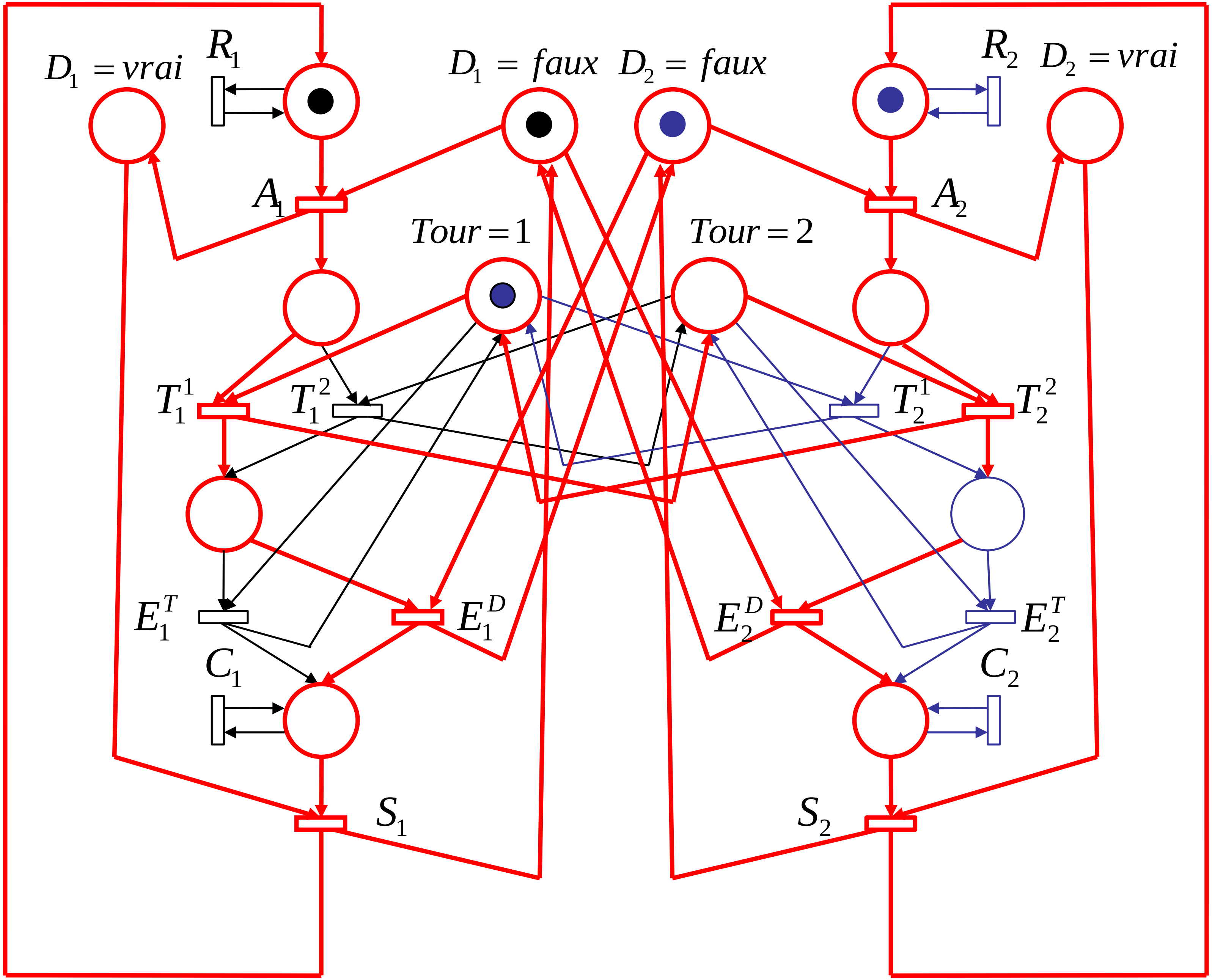



T-invariant 5

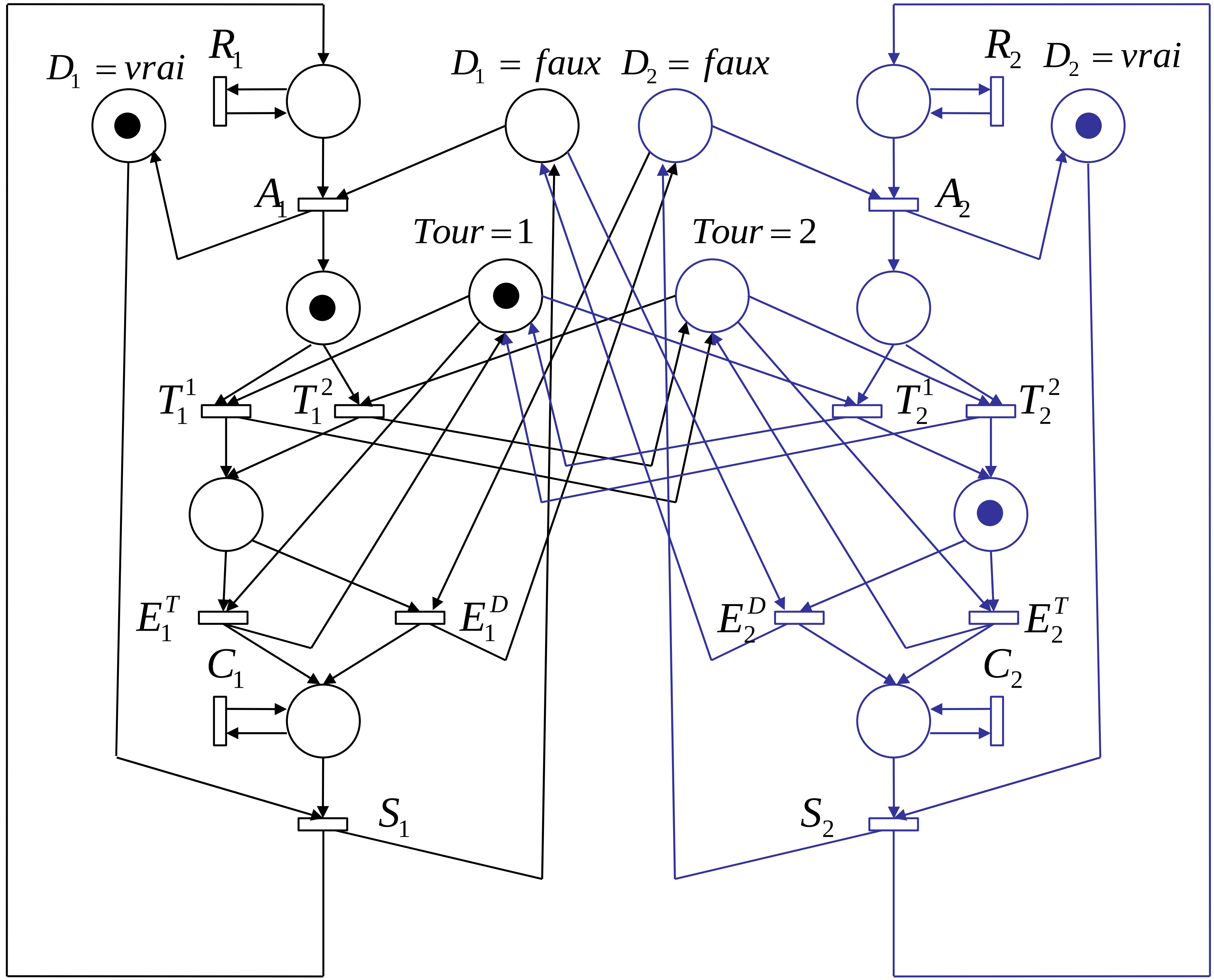



T-invariant 5 : retour à l'état initial

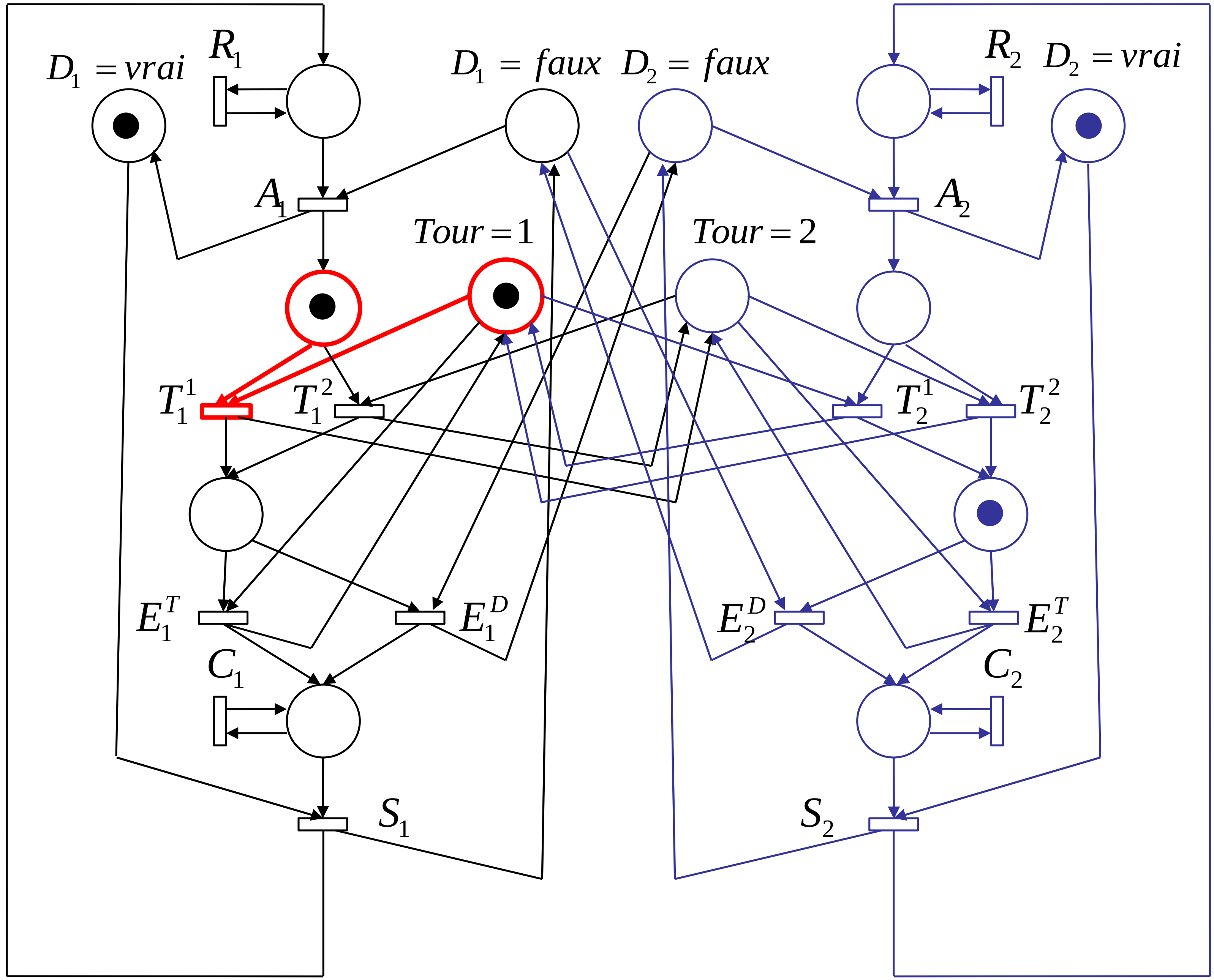
J.-M. Delosme - SED



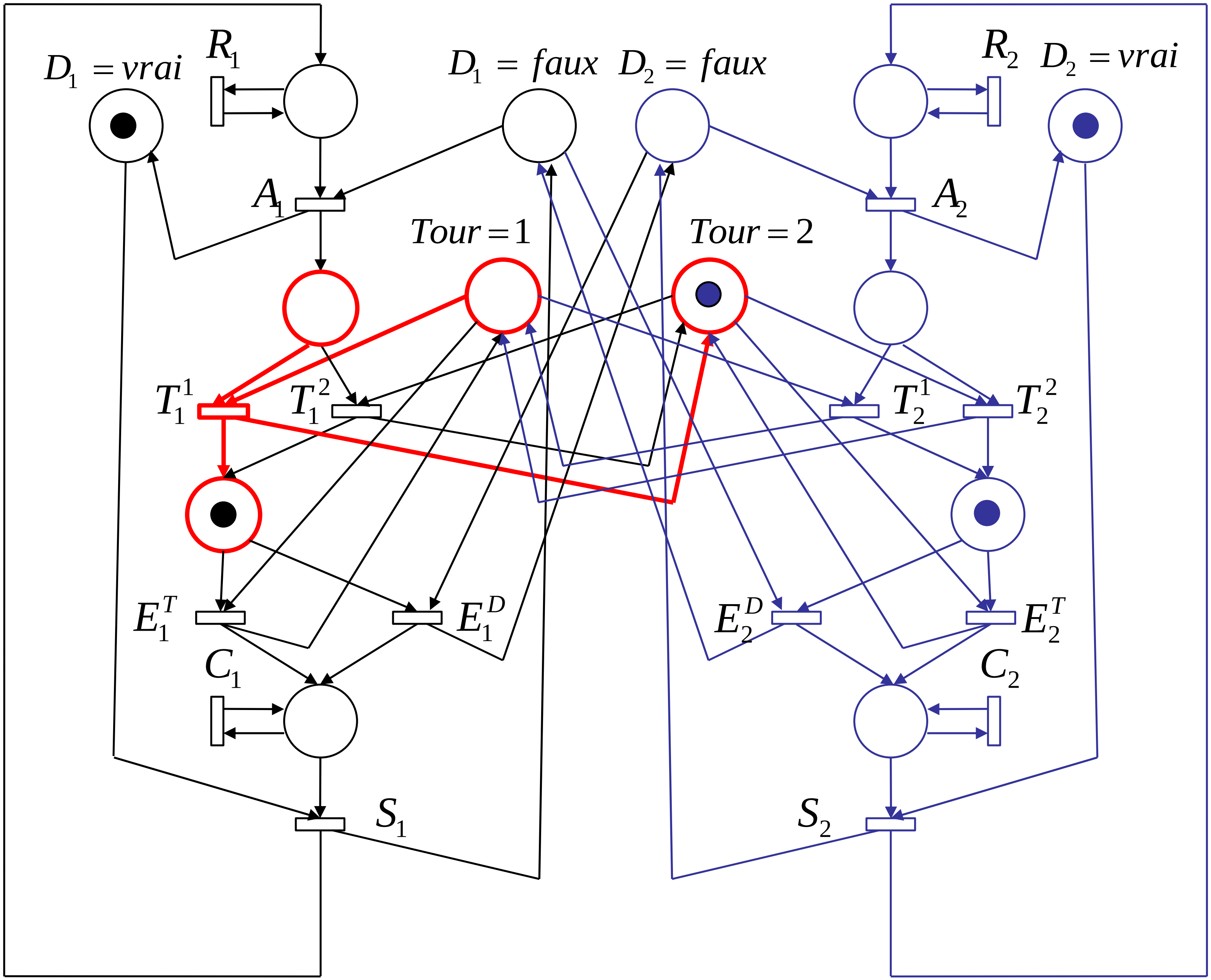

T-composant 5



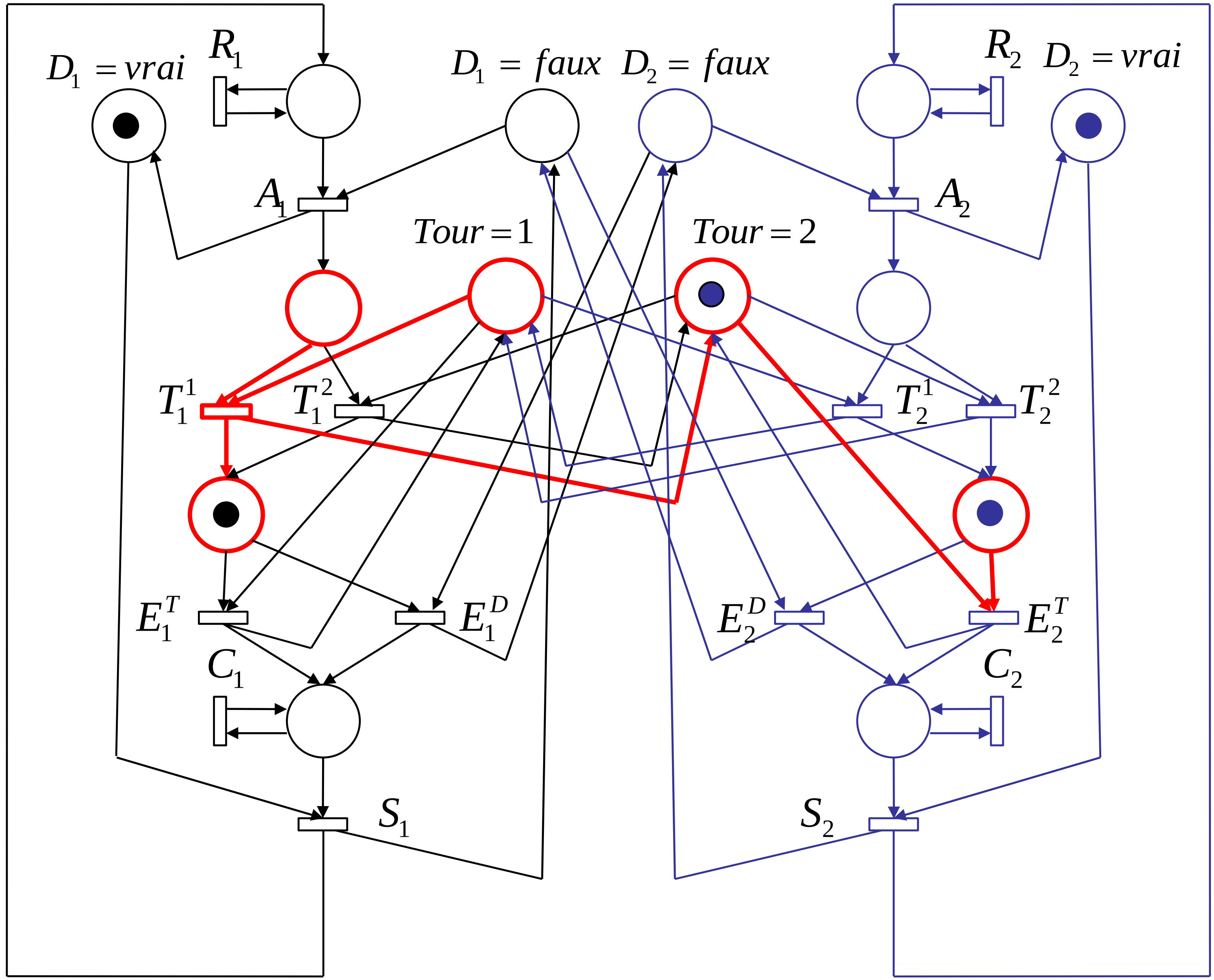

Processus 2 en attente de sa section critique



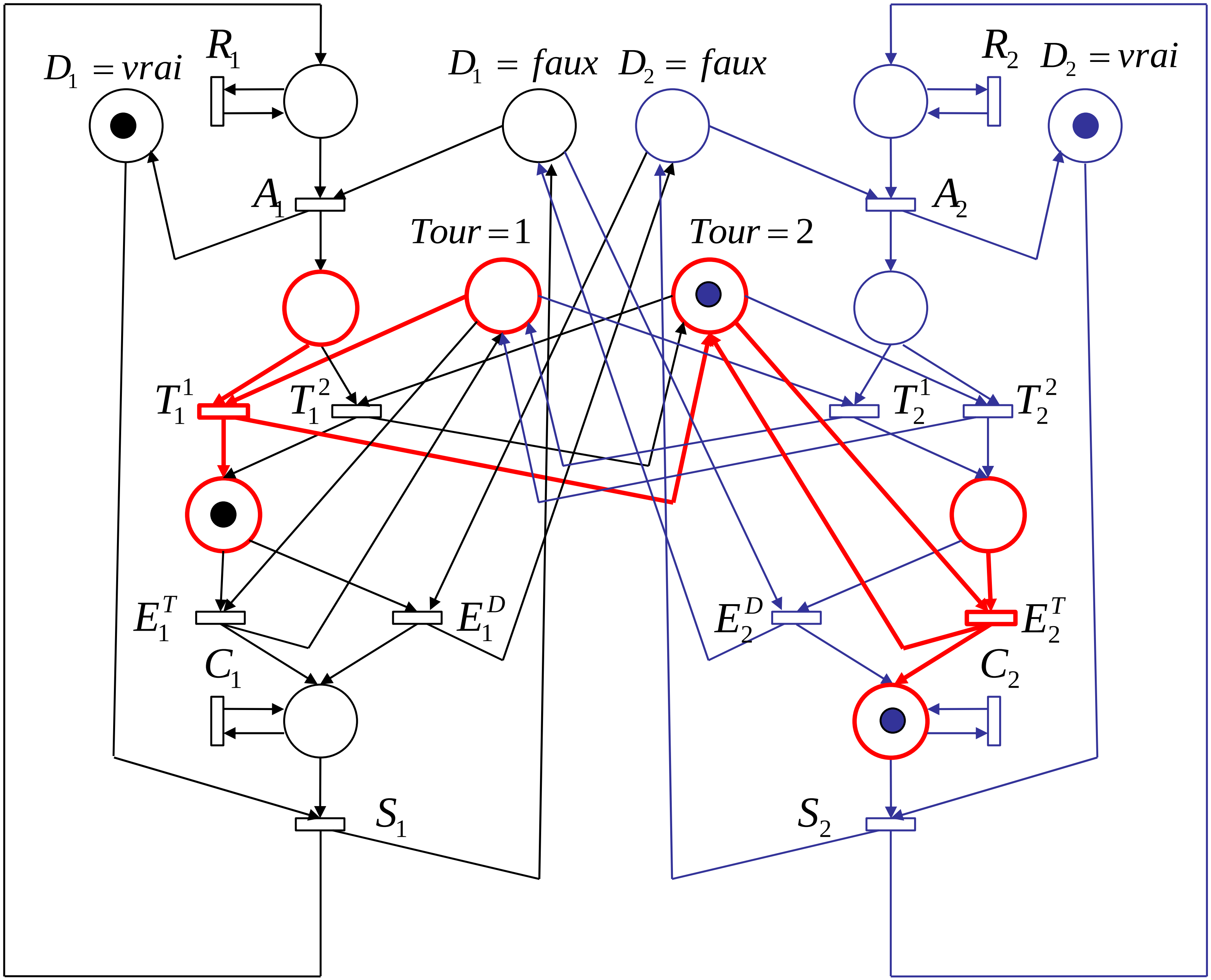

J.-M. Delosme - SED



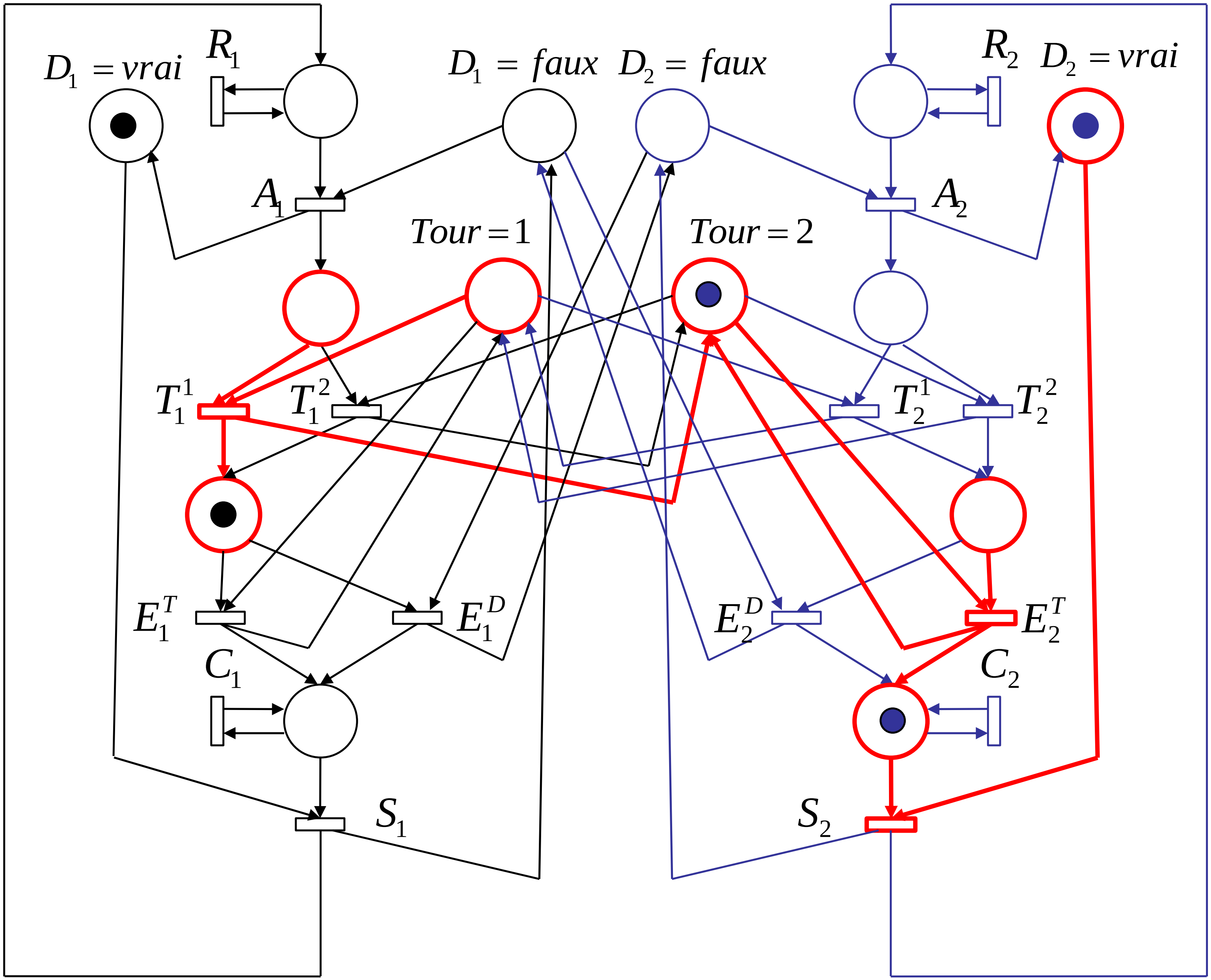

Processus 1 aussi en attente de sa section critique



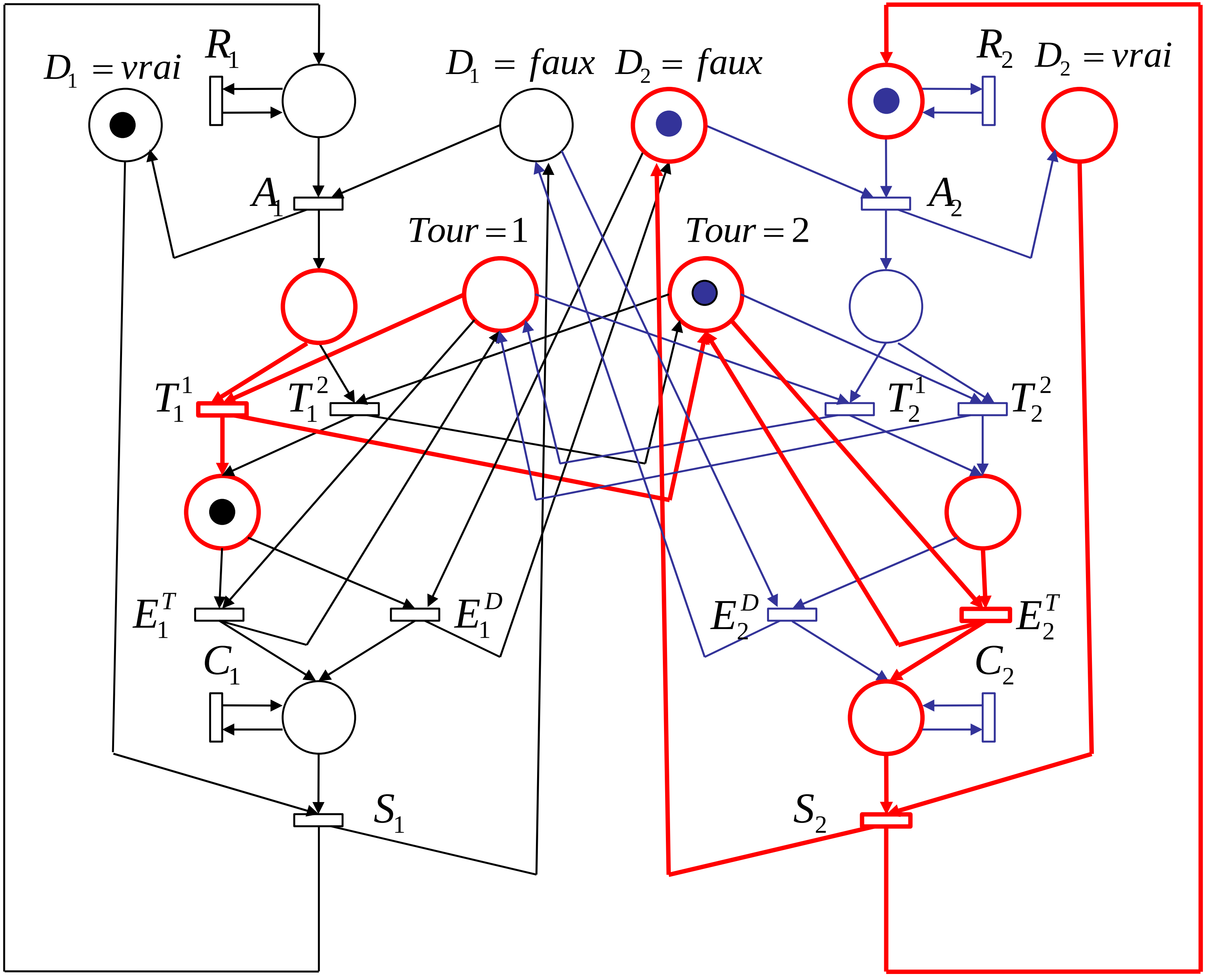

J.-M. Delosme - SED



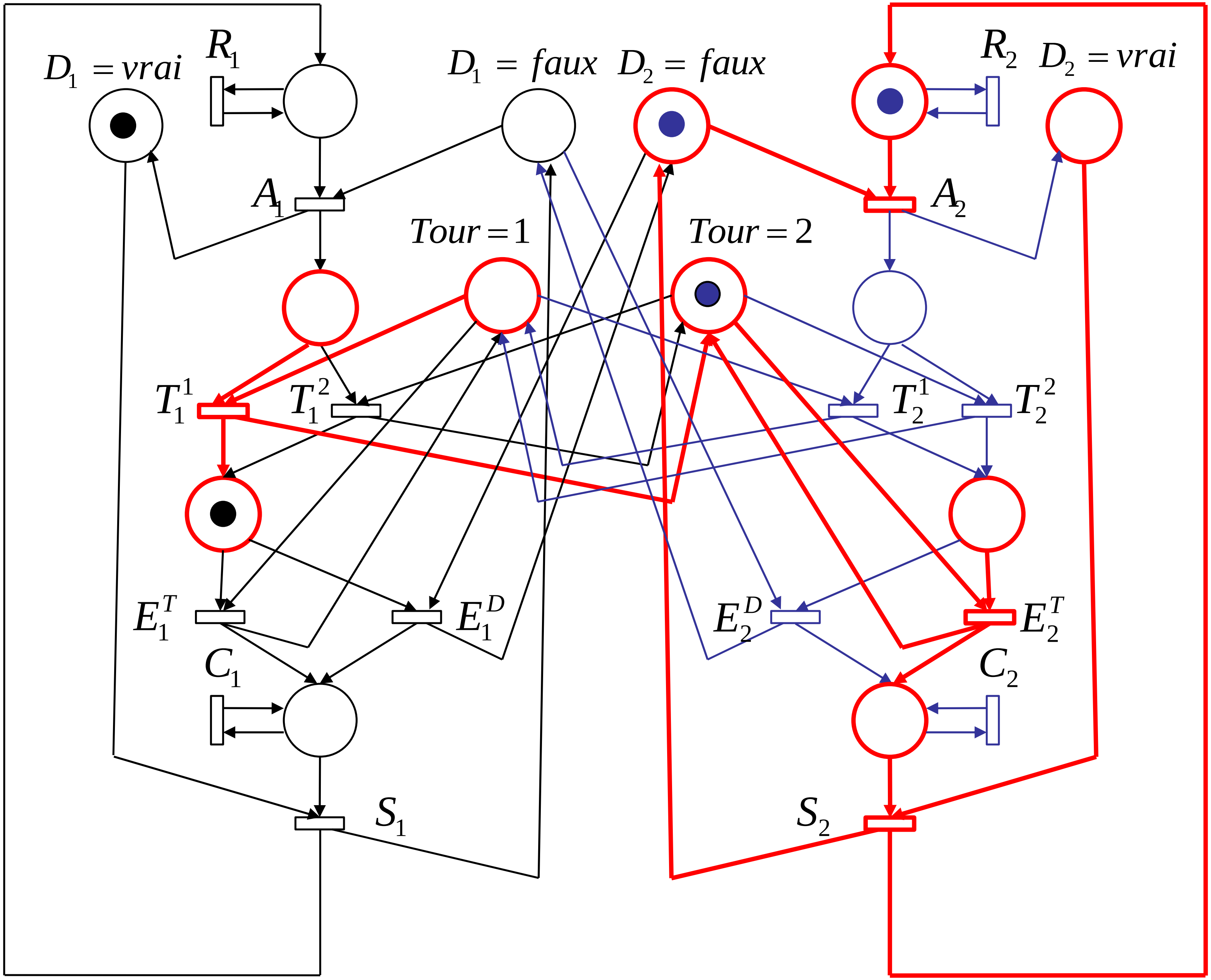

Processus 1 en attente de sa section critique



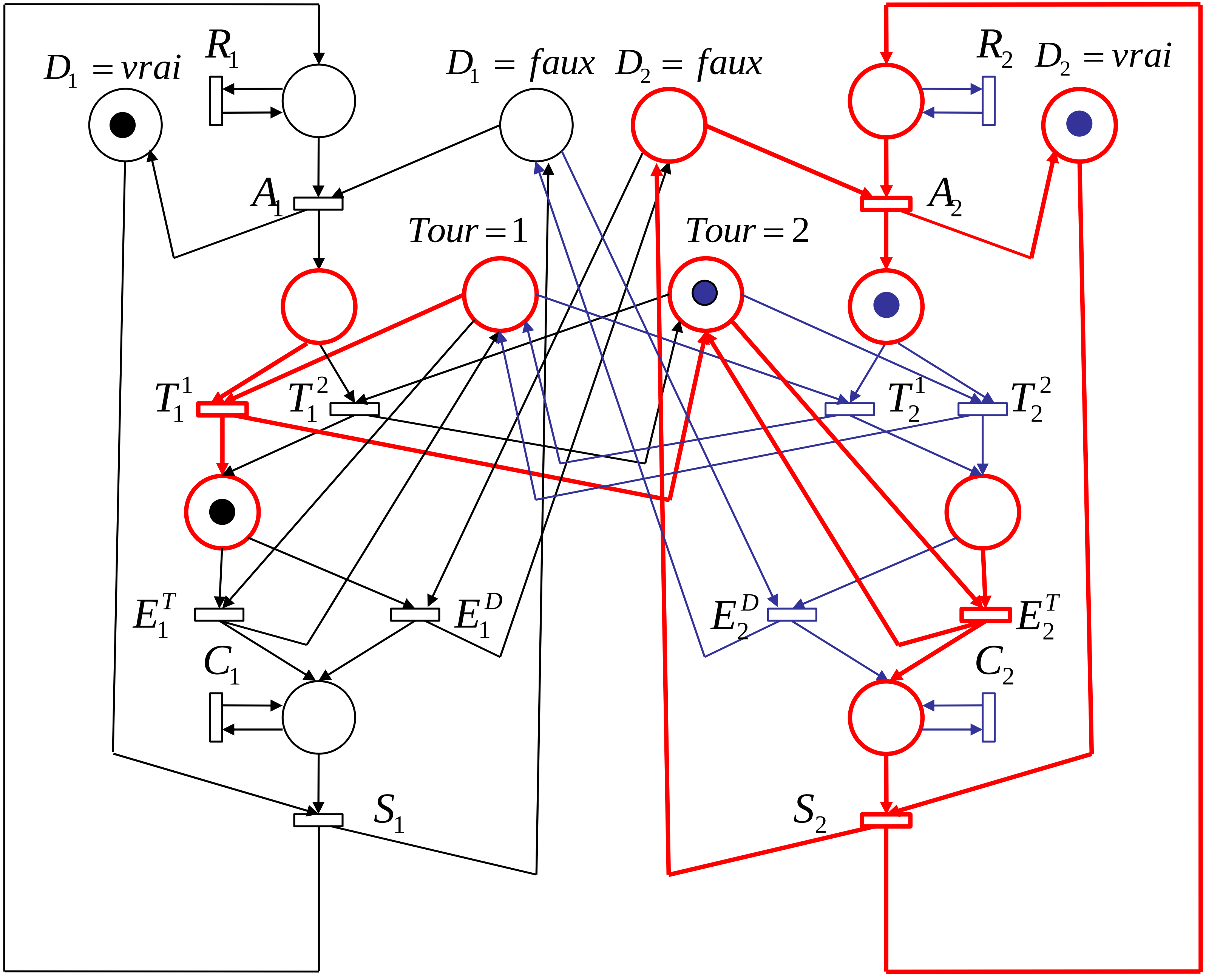

J.-M. Delosme - SED



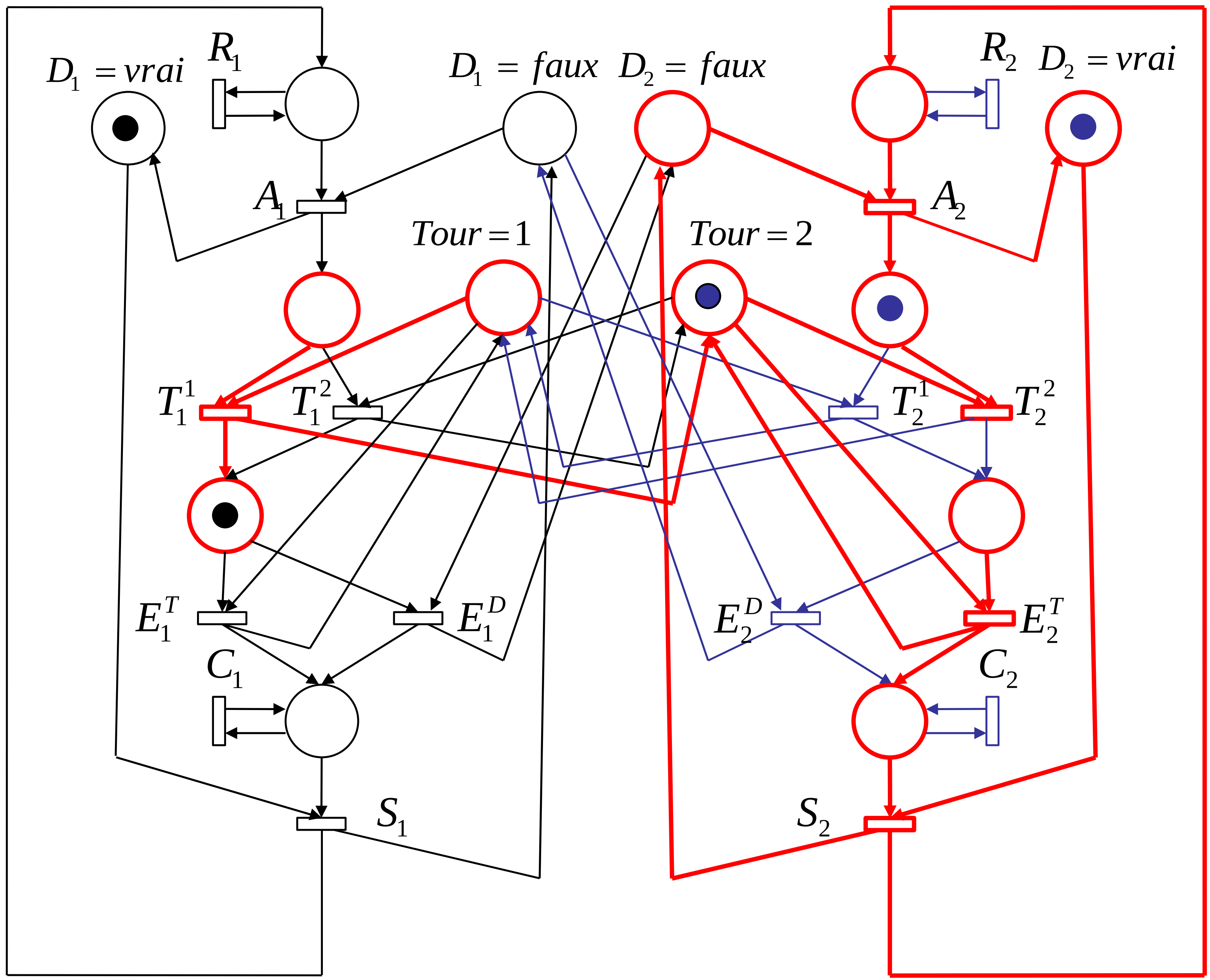

J.-M. Delosme - SED



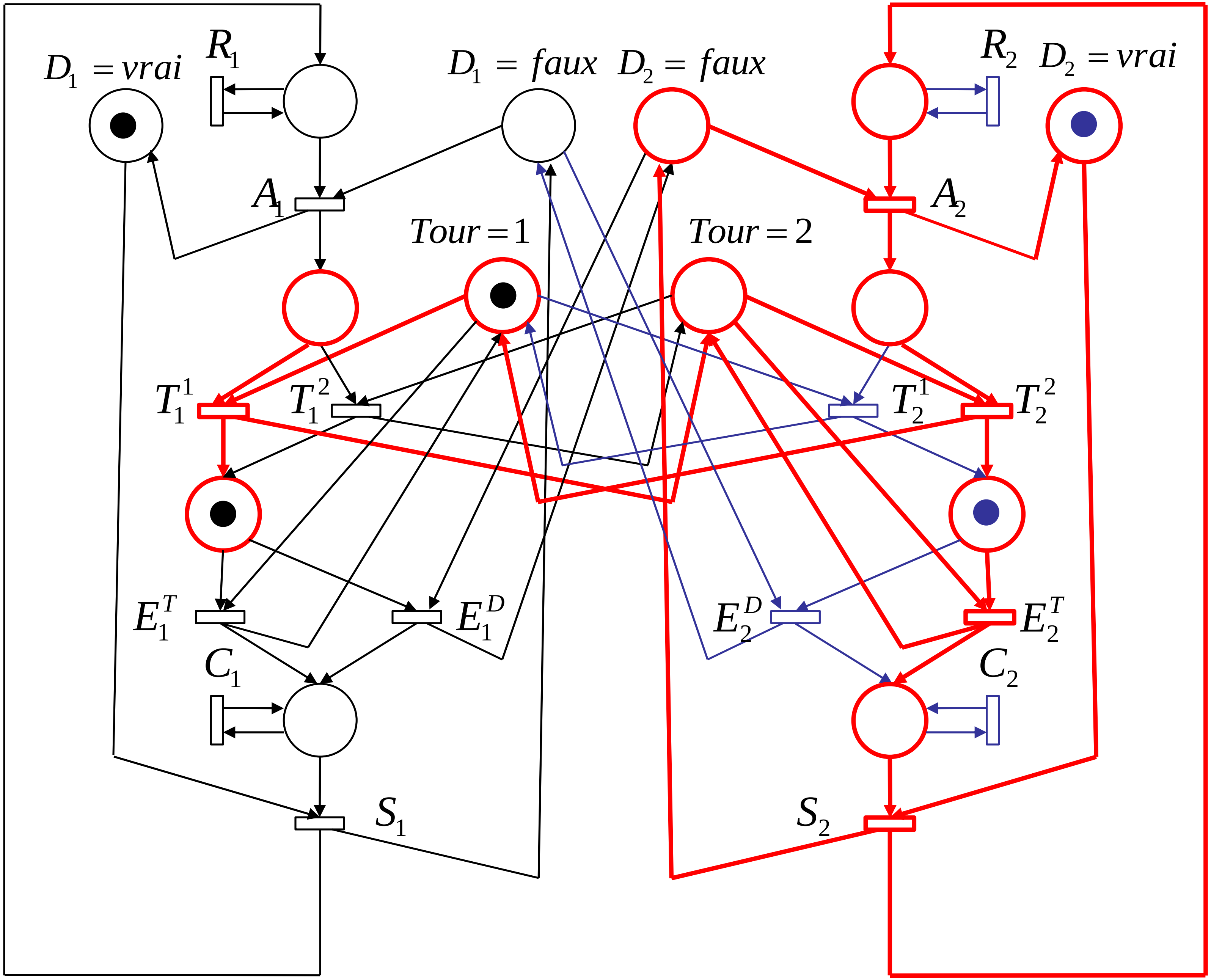

J.-M. Delosme - SED




J.-M. Delosme - SED

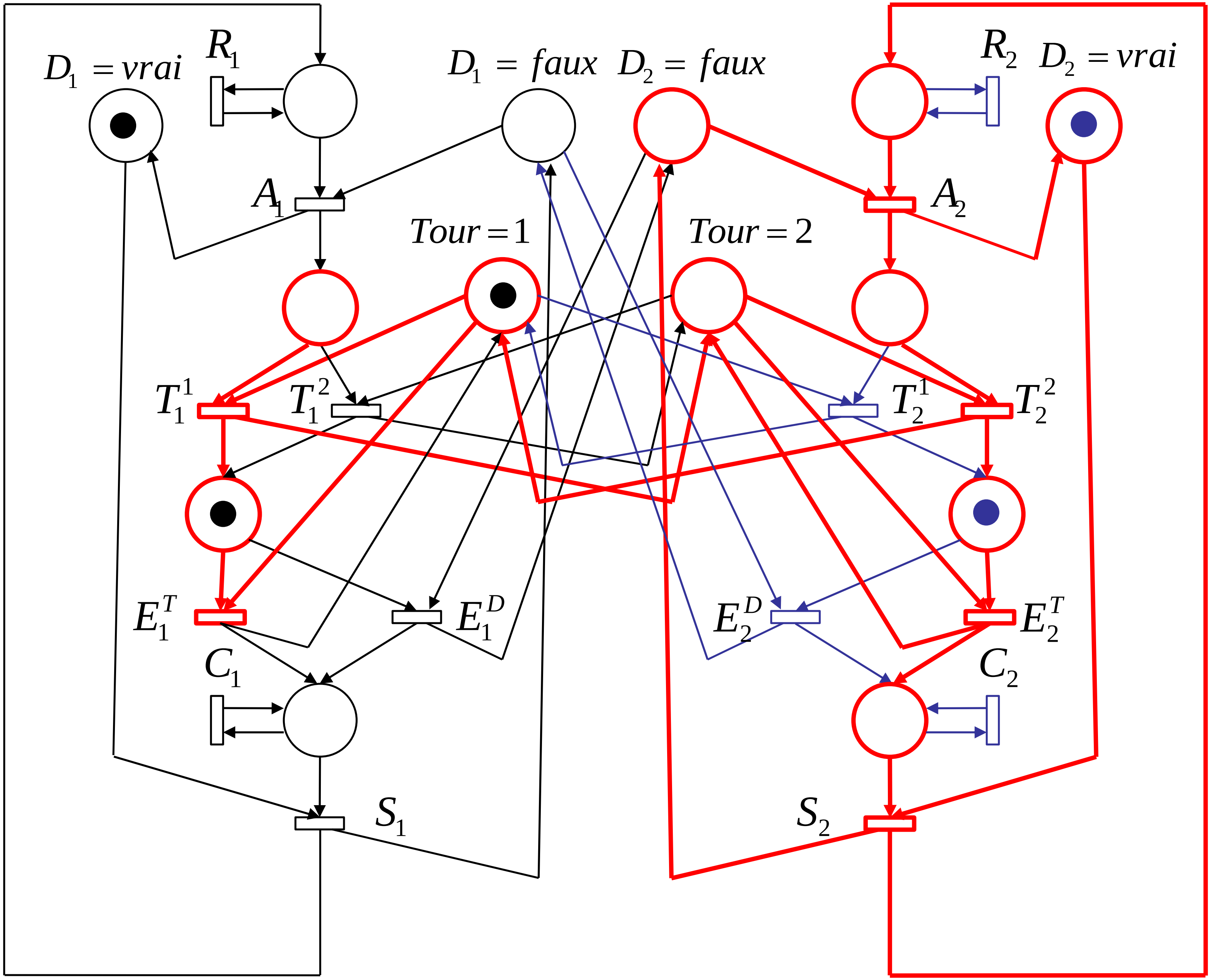



J.-M. Delosme - SED

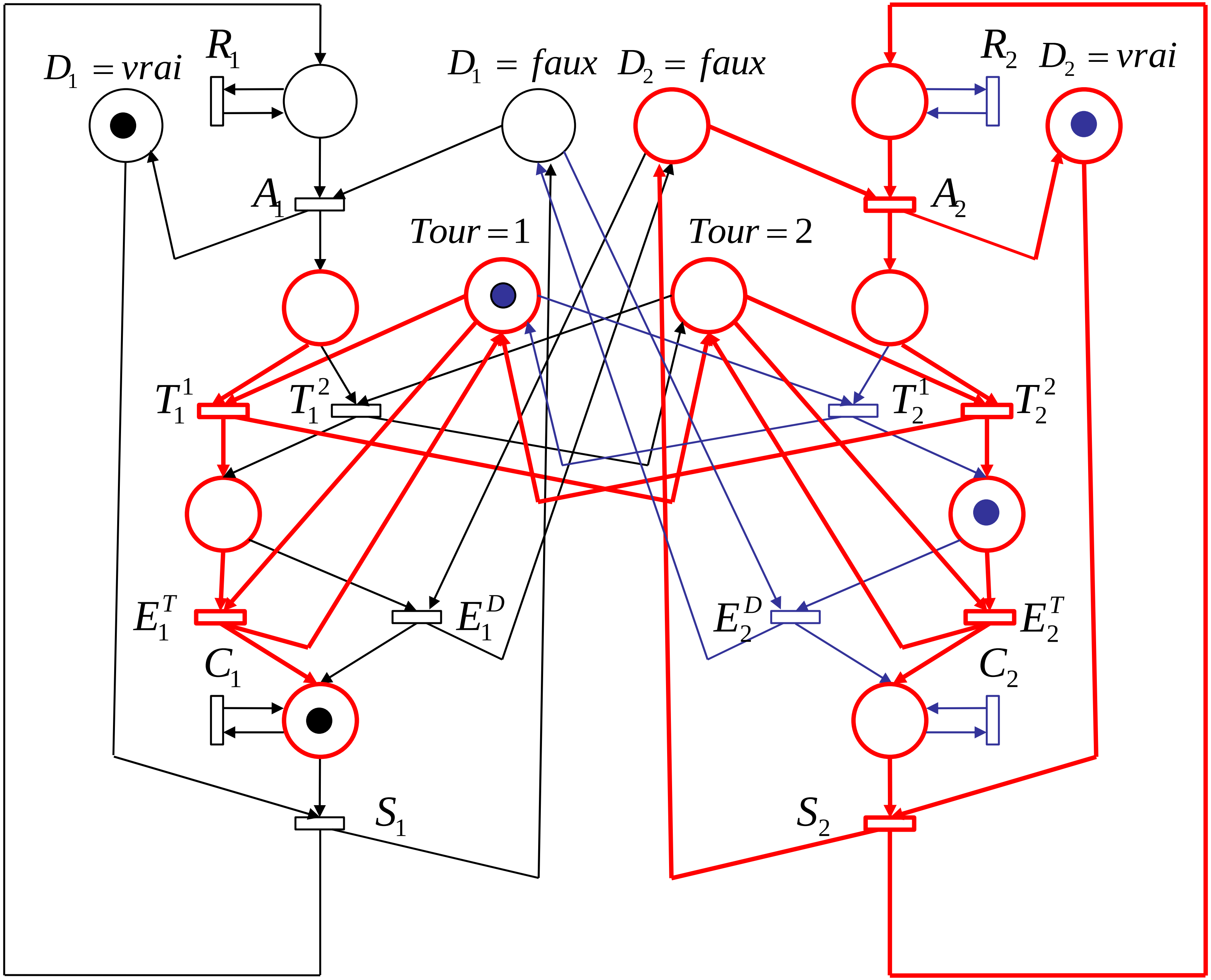



Processus 2 aussi en attente de sa section critique

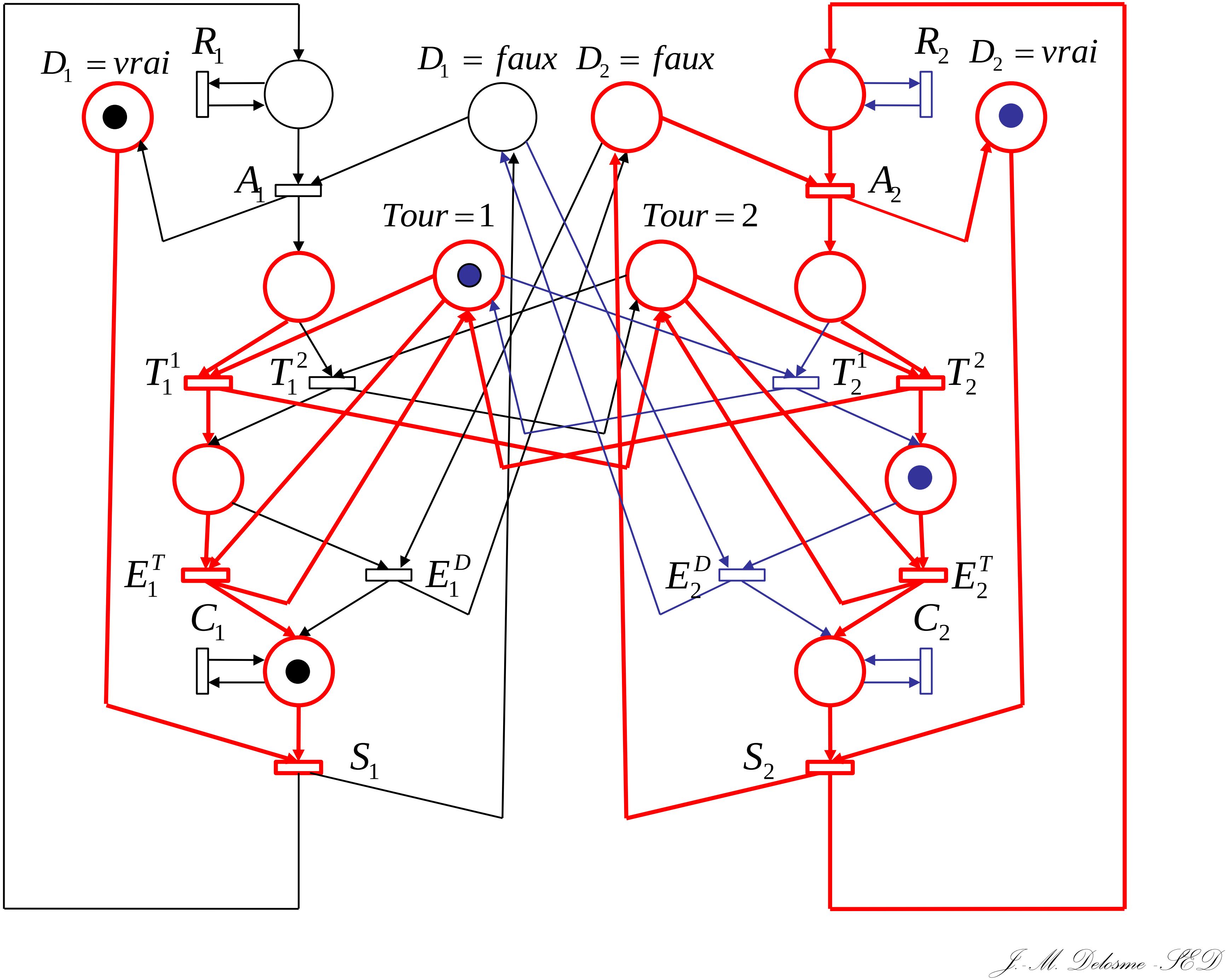
J.-M. Delosme - SED

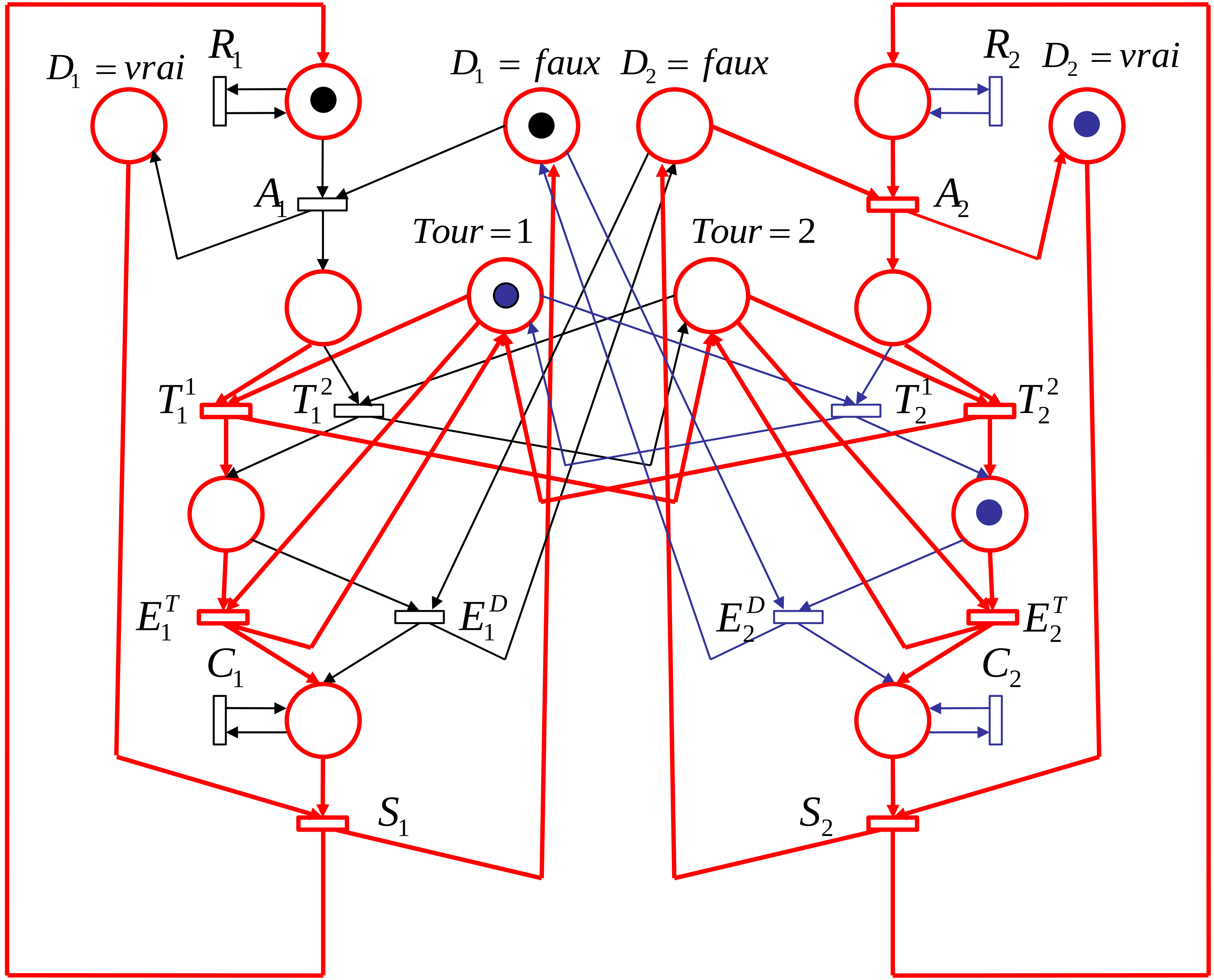



J.-M. Delosme - SED

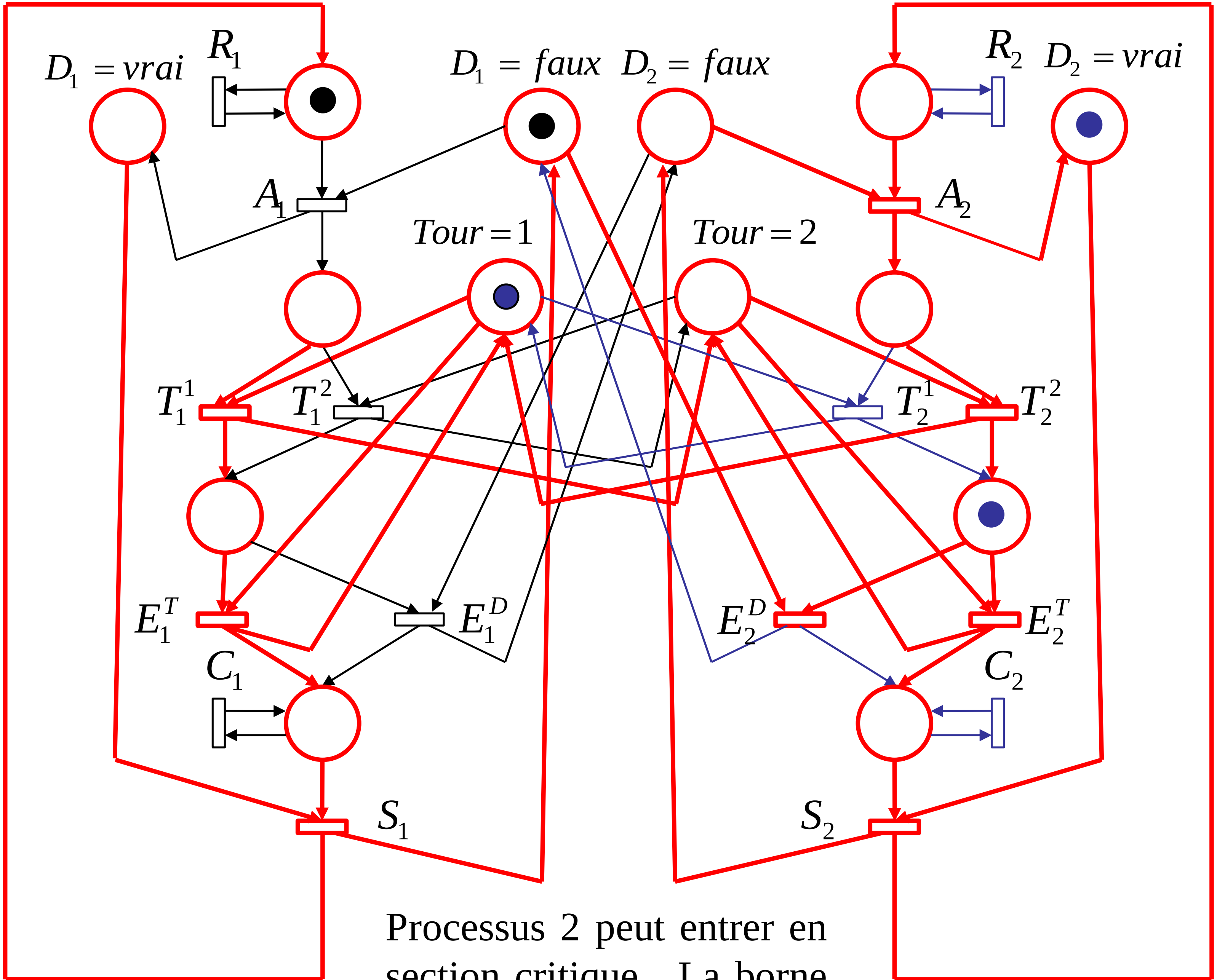



Processus 2 en attente de sa section critique


$D_1 = vrai$
$R_1$
$D_1 = faux$
$D_2 = faux$
$R_2$
$D_2 = vrai$
$A_1$
$A_2$
$Tour = 1$
$Tour = 2$
$T_1^1$
$T_1^2$
$T_2^1$
$T_2^2$
$E_1^T$
$E_1^D$
$E_2^D$
$E_2^T$
$C_1$
$C_2$
$S_1$
$S_2$
J.-M. Delosme - SED




J.-M. Delosme - SED




Processus 2 peut entrer en section critique. La borne est donc égale à 1.

J.-M. Delosme - SED

Y-a-t-il d'autres T-invariants ?

$$
\begin{array}{c}
A_1 \\ T_1^1 \\ T_1^2 \\ E_1^T \\ E_1^D \\ S_1 \\ A_2 \\ T_2^1 \\ T_2^2 \\ E_2^T \\ E_2^D \\ S_2
\end{array}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

Oui. On peut substituer $E_1^T$ à $E_1^D$ et/ou $E_2^T$ à $E_2^D$ dans le T-composant 5, comme on l'a fait dans les T-composants 1 et 3 pour obtenir les T-composants 2 et 4. Clairement les T-invariants ainsi obtenus ne seront pas indépendants de ceux définis par $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ et $\boldsymbol{u}_5$.

$$\boldsymbol{u}_1 = [1, 0, 1, 0, \boxed{1}, 1, 0, 0, 0, 0, 0, 0]$$
$$\boldsymbol{u}_2 = [1, 0, 1, \boxed{1}, 0, 1, 0, 0, 0, 0, 0, 0]$$
$$\boldsymbol{u}_3 = [0, 0, 0, 0, 0, 0, 1, 1, 0, 0, \boxed{1}, 1]$$
$$\boldsymbol{u}_4 = [0, 0, 0, 0, 0, 0, 1, 1, 0, \boxed{1}, 0, 1]$$
$$\boldsymbol{u}_5 = [1, \boxed{1}, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1]$$

Ces 5 vecteurs sont indépendants.
Ils définissent des combinaisons linéaires des lignes de $B$ égales à 0.

D'après l'étude des P-invariants (7 P-invariants indépendants) le rang des colonnes de la matrice $B$ est égal à 7. En effet
rang des colonnes $=$ nb. de colonnes – nb. de combinaisons linéaires indépendantes des colonnes égales à 0 $= 14 - 7$.

Le rang des lignes d'une matrice est égal au rang de ses colonnes.
Le rang des lignes de la matrice $B$ est donc égal à 7. Comme
rang des lignes $=$ nb. de lignes – nb. de combinaisons linéaires indépendantes des lignes égales à 0, on a
nb. de combinaisons linéaires indépendantes des lignes égales à 0 = nb. de lignes – rang des lignes $= 12 - 7 = 5$.
Il n'y a donc pas de T-invariant qui serait indépendant des 5 que l'on a trouvés.