Abrupt Behaviour Changes in Cellular Automata under Asynchronous Dynamics

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Abstract. We propose an analysis of the relaxation time of the elementary finite cellular automaton **214** (Wolfram coding) under α-asynchronous dynamics (i.e. each cell independently updates with probability $0 < \alpha \leqslant 1$ at each time step).
While cellular automata have been intensively studied under synchronous dynamics (all cells update at each time step), much less work is available about asynchronous dynamics. In particular, the robustness to asynchronism is a feature which is far from being cleared up.
In a classifying attempt, Fates *et al* [2] have listed all the double-quiescent elementary cellular automata under fully asynchronous dynamics (one single cell is updated at each time step) according to their relaxation time. This mathematical analysis confirmed the behaviours observed by simulation, truly different from the synchronous dynamics. In a sequel paper [1], they extended their analysis to this class of automata under α-asynchronous dynamics. Moreover they exhibit new phenomena which are impossible under fully asynchronous dynamics, the global behaviour of most of the automata is the same under both α-asynchronism and full asynchronism. However unlike [2], they did not complete the whole classification of relaxation times and left some conjectures concerning four automata, among which automaton **214** which seems to have a specific behavior under α-synchronous dynamics. Our work partially answers one of those conjectures, and both illustrates the richness of the behaviours involved by asynchronism on cellular automata and the challenge of their mathematical prediction. Far from being a marginal case study, our analysis provides a very relevant example of the way the dynamics is affected by asynchronism and of the mathematical tools which can be used to predict the asymptotic behaviour of such complex models.

1 Introduction

The aim of this article is to analyze the asynchronous behavior of the unbounded finite cellular automaton **214**. Cellular automata are widely used to model systems involving a huge number of interacting elements such as agents in economy, particles in physics, proteins in biology, distributed systems, etc. In most of these applications, in particular in many real system models, agents are not synchronous. Depending on the transition rules, the behaviour of the system may vary widely when asynchronism increases in the dynamics. More generally one can ask how much does asynchronous in real system perturbs computation. In spite of this lack of synchronism, real living systems are very resilient over time. One might then expect the cellular automata used to model these systems to be robust to asynchronism and to other kind of failure as well (such as misreading the states of the neighbors). It turns out that the resilience to asynchronism widely varies from one automata to another (e.g., [3,4]). Only few theoretical studies exist on the influence of asynchronism. Most of them usually focus on one specific cellular automata (e.g., [5,6,7]) and do not address the problem globally. In 2003, Gács shows in [8] that it is undecidable to determining if in a given automota, the sequences of changes of states followed by a given cell is independent of the history of the updates. Related work on the existence of stationary distribution on infinite configurations for probabilistic automata can be found in [9].

We continue here a study begun in [2] and [1] on the effects of asynchrony on the global evolution of the system given an *arbitrary* set of local rules, and in particular how does asynchronicity affects its relaxation time. In [2], the authors carried out a complete analysis of the class of one-dimensional double quiescent elementary cellular automata (DQECA), where each cell has two states `0` and `1` which are quiescent (i.e., where each cell for which every cell in its neighbourhood is in the same state, remains in the same state) and where each cell updates according to its state and the states of its two immediate neighbours. They study the behaviour of these automata under fully asynchronous dynamics, where only one random cell is updated at each time step. They show that one can classify the 64 DQECAs in six categories according to their relaxation times under full asynchronism (either constant, logarithmic, linear, quadratic, exponential or infinite) and furthermore that the relaxation time characterizes their behaviour, i.e., that all automata with equivalent relaxation times present the same kind of space-time diagrams. In [1], this study is extended to a continuous range of asynchyronism from fully asynchronous dynamics to fully synchronous dynamics: the α*-asynchronous dynamics*, with $0 < \alpha \leqslant 1$. In this setting, each cell is updated independently with probability α at each time step. When α varies from 1 down to 0, the α-asynchronous dynamics evolves from the fully synchronous regime to a more and more asynchronous regime. As α approaches 0, the probability that the updates involve at most one cell tends to 1, and the dynamics gets closer and closer to a kind of fully asynchronous dynamics up to a time rescaling by a factor $1/\alpha$.

The comparison between the fully asynchronous dynamics and the synchronous dynamics in [2] shows that most of the studied automata have drastically different behaviors. The comparison between the fully asynchronous dynamics and the α-asynchronous dynamics in [1] shows that new phenomena could appear under α-asynchronous dynamics. Nevertheless after rescaling of the time, most of the studied automata seem to have the same global behavior under these two dynamics. The only automata where these phenomena change drastically its behavior is automaton **194**. Its relaxation time is $O(n^3)$ under fully asynchronous dynamics, $O(\frac{n}{\alpha^2(1-\alpha)})$ under α-asynchronous dynamics and it diverge under synchronous dynamics. Thus there is a speed up between fully asynchronous and α-asynchronous dynamics because of a so called *spawning* phenomenon (see [1]). The authors conjecture that four other automata have a specific behavior under α-asynchronous dynamics. Cellular automaton **214** studied here is one of them. It diverges (i.e., it never reaches a fixed point) under both fully asynchronous dynamics and synchronous dynamics. Nevertheless, we prove here that cellular automaton **214** converges to a fixed point in linear time under α-asynchronous dynamics when $\alpha > 0.9999$ and we also exhibit the phenomenon accountable for this fast convergence. Now, this is the most explicit case to show the difference between α-asynchronous dynamics and the two other dynamics.

Section 2 introduces the main definitions and presents our main result. Section 3 presents the probabilistic tools developed in [1] that will be used for the analysis in section 5. Section 4 gives the intuition behind the neighborhood masks tree.

2 Definitions, Notations and Main Results

In this paper, we consider the elementary cellular automaton **214** on finite size configurations with periodic boundary conditions. See [1] for complete definitions.

Definition 1. *An* Elementary Cellular Automata (ECA) *is given by its* transition function $\{\delta : \{0,1\}^3 \to \{0,1\}\}$. *We denote by* $Q = \{0,1\}$ *the set of* states.

We denote by $U = \mathbb{Z}/n\mathbb{Z}$ *the set of* cells. *A* finite configuration *with periodic boundary conditions* $x \in Q^U$ *is a word indexed by* U *with letters in* Q.

Definition 2. *For a given pattern* $w \in Q^*$, *we denote by* $|x|_w = \#\{i \in U : x_{i+1} \dots x_{i+|w|} = w\}$ *the number of occurrences of* w *in configuration* x.

Definition 3. Here is the transition function of cellular automaton **214**:

$x\ y\ z$	000	001	100	101	010	011	110	111
$214(x,y,z)$	0	1	1	0	1	0	1	1

We consider three kinds of dynamics for ECAs: the *synchronous dynamics*, the *α-asynchronous dynamics* and the *fully asynchronous dynamics*. The synchronous dynamics is the classic dynamics of cellular automata, where the transition function is applied at each (discrete) time step on each cell simultaneously.

Definition 4 (Synchronous Dynamics). *The* synchronous dynamics $S_\delta : Q^U \to Q^U$ *of an ECA* δ, *associates deterministically to each configuration* x *the configuration* y, *such that for all* $i \in U$, $y_i = \delta(x_{i-1}, x_i, x_{i+1})$.

Definition 5 (Asynchronous Dynamics). *An* asynchronous dynamics AS_δ *of an ECA* δ *associates to each configuration* x *a random configuration* y, *such that* $y_i = x_i$ *for* $i \notin S$, *and* $y_i = \delta(x_{i-1}, x_i, x_{i+1})$ *for* $i \in S$, *where* S *is a random subset of* U *chosen by a daemon. We consider two types of asynchronous dynamics:*

– *in the* α-asynchronous dynamics*, the daemon selects at each time step each cell i in S independently with probability α where $0 < \alpha \leqslant 1$. The random function which associates the random configuration y to x according to this dynamics is denoted AS_δ^α.*
– *in the* fully asynchronous dynamics*, the daemon chooses a cell i uniformly at random and sets $S = \{i\}$. The random function which associates the random configuration y to x according to this dynamics is denoted AS_δ^F.*

For a given ECA δ, we denote by x^t the random variable for the configuration obtained after t applications of the asynchronous dynamics function AS_δ on configuration x, i.e., $x^t = (AS_\delta)^t(x)$.

Definition 6 (Fixed point). *We say that a configuration x is a* fixed point *for δ under asynchronous dynamics if $AS_\delta(x) = x$ whatever the choice of S is (the cells to be updated). $\mathfrak{F}_\delta$ denotes the set of fixed points for δ.*

Fact 1 *if n is even then $\mathfrak{F}_{214} = \{\mathtt{0}^n, \mathtt{1}^n, (\mathtt{01})^{n/2}\}$, otherwise $\mathfrak{F}_{214} = \{\mathtt{0}^n, \mathtt{1}^n\}$. The configuration $\mathtt{0}^n$ cannot be reached from any other configurations whatever the dynamics is.*

The set of fixed points for the considered asynchronous dynamics is clearly identical to $\{x : S_\delta(x) = x\}$ the set of fixed points of the synchronous dynamics.

Definition 7 (Relaxation Time). *Given an ECA δ and a configuration x, we denote by $T_\delta(x)$ the random variable for the time elapsed until a fixed point is reached from configuration x under an asynchronous dynamics, i.e., $T_\delta(x) = \min\{t : x^t \in \mathfrak{F}_\delta\}$. The* relaxation time *of ECA δ is $\max_{x \in Q^U} \mathbb{E}[T_\delta(x)]$.*

We can now state our main theorem.

Theorem 2 (Main result). *Under α-asynchronous dynamics when $\alpha > 0.9999$, the relaxation time T_{214} of cellular automaton **214** is $O(\frac{n}{1-\alpha})$.*

3 Lyapunov functions based on local neighbourhoods

The reader may find more detailed definitions in [1].

Definition 8 (Mask). *A* mask $\dot{m}$ *is a word on $\{\mathtt{0}, \mathtt{1}, \dot{\mathtt{0}}, \dot{\mathtt{1}}\}$ containing exactly one dotted letter in $\{\dot{\mathtt{0}}, \dot{\mathtt{1}}\}$. We say that the cell i in configuration x matches the mask $\dot{m} = m_{-k} \dots m_{-1} \dot{m}_0 m_1 \dots m_l$ if $x_{i-k} \dots x_i \dots x_{i+l} = m_{-k} \dots m_0 \dots m_l$. We denote by m the undotted word $m_{-k} \dots m_0 \dots m_l$.*

Notation 1 *In the next sections, some letters of a mask will receive a label $\times$ or $?$. An unlabeled cell won't change its state at the next time step even if it updates. An $\times$-labeled will change its state at the next time if it updates. If the cell is labeled $?$, we don't have enough information in the mask to decide if the cell may change its state at the next time step.*

Definition 9 (Masks basis). *A* masks basis $\mathcal{B}$ *is a finite set of masks such that for any configuration x and any cell i, there exists an unique $\dot{m} \in \mathcal{B}$ that matches cell i.*

A masks basis $\mathcal{B}$ can be represented by a binary tree where the masks of $\mathcal{B}$ are the labels of the leaves.

Masks bases will be used to define Lyapunov weight functions from local patterns. It provides an efficient tool to validate exhaustive case analysis.

Definition 10 (Local weight function). *A* local weight function f *is a function from a masks basis $\mathcal{B}$ to $\mathbb{Z}$. The local weight of the cell i in configuration x given by f is $F(x,i) = f(\dot{m})$ where $\dot{m}$ is the unique mask in $\mathcal{B}$ matching cell i. The* weight of a configuration x *given by f is defined as $F(x) = \sum_i F(x,i)$.*

Fact 3 *Given a local weight function $f : \mathcal{B} \to \mathbb{Z}$, the weight of configuration x is equivalently defined as: $F(x) = \sum_{\dot{m}\in\mathcal{B}} f(\dot{m}) \cdot |x|_m$.*

Notation 2 *For a given random sequence of configurations $(x^t)_{t\in\mathbb{N}}$ and a weight function F on the configurations, we denote by $(\Delta F(x^t))_{t\in\mathbb{N}}$ the random sequence $\Delta F(x^t) = F(x^{t+1}) - F(x^t)$.*

The next lemma provides upper bounds on stopping times for the markovian sequence of configurations $(x^t)_{t\in\mathbb{N}}$ subject to a weight function F decreasing or remaining constant on average (a *Lyapunov function*). Its proof can be found in [2].

Lemma 1. *Let $m \in \mathbb{Z}_+$ and $\epsilon > 0$. Consider (x^t) a random sequence of configurations, and F a weight function such that $(\forall x)\ F(x) \in \{0,\dots,m\}$. Assume that if $F(x^t) > 0$, then $\mathbb{E}[\Delta F(x^t)|x^t] \leqslant -\epsilon$. Let $T = \min\{t : F(x^t) = 0\}$ denote the random variable for the first time t where $F(x^t) = 0$. Then, $\mathbb{E}[T] \leqslant \frac{m+F(x^0)}{\epsilon}$.*

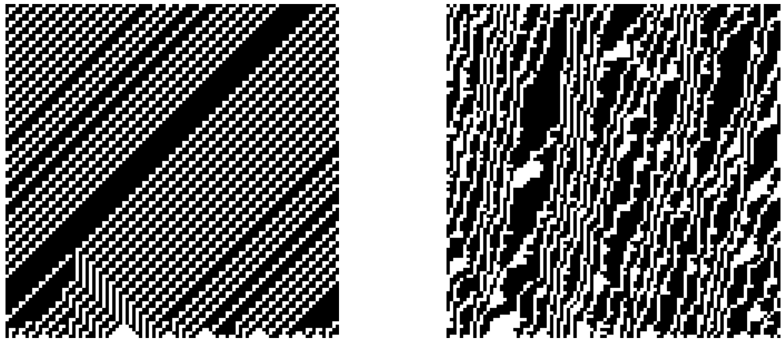
4 Informal description of 214's behaviour

Figure 1 shows the automaton **214** under asynchronous and fully asynchronous dynamics.

Under fully asynchronous dynamics a configuration cannot reached a fixed point from a non fixed point configuration. In fact under fully asynchronous dynamics, the number of regions (which is also $|x^t|_{\texttt{10}}$ or $|x^t|_{\texttt{01}}$) cannot increase and it decreases only when a cell is updated in the neighborhood `010` of `101`. So the number of regions is constant for a configuration evolving under fully asynchronous dynamics of automaton **214**.

Now we consider the α-asynchronous dynamics as shown in figure 2.

First, as shown in figure 2(b) the automaton may converge to the fixed point $\texttt{0}^n$. According to simulations, the relaxation time appears to be linear of the size of n and is conjectured to be $O(\frac{n}{\alpha^2(1-\alpha)})$. Second, the number of regions can increase or decrease because of two new phenomena that have already



(a) Synchronous dynamics (b) Totally asynchronous dynamics

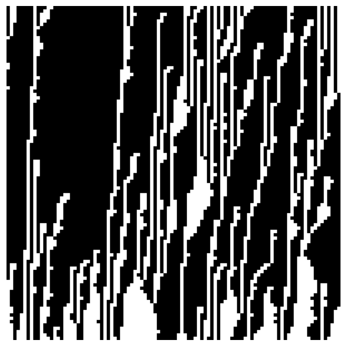
Fig. 1. BCF under different dynamics

be observed in [1]: the *spawning* phenomenon and the *annihilation* phenomenon (see fig. 3). Indeed a pattern `1001` may evolve to `1111` (the number of regions decrease) and a pattern `0011` may evolve to `0101` (the number of regions increase). Thus the only way to decrease the number of regions is the annihilation phenomenon. So the key pattern is `1001`. Unfortunately, if we consider the evolution pattern `10011`:

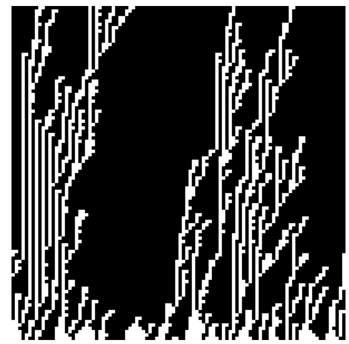
with probability	pattern `1`$\underset{\times}{\texttt{0}}\underset{\times}{\texttt{0}}\underset{\times}{\texttt{1}}$`1` evolves to	distance to fixed point seems to
α^3	`11101`	not vary
$\alpha^2(1-\alpha)$	`11001`	not vary
$\alpha^2(1-\alpha)$	`10101`	increase
$\alpha^2(1-\alpha)$	`11111`	decrease
$\alpha(1-\alpha)^2$	`10111`	not vary
$\alpha(1-\alpha)^2$	`11011`	not vary
$\alpha(1-\alpha)^2$	`10001`	increase
$(1-\alpha)^3$	`10011`	not vary

The difficulty of the proof is that a `0`-region in the pattern `10011` has the same probability, $\alpha^2(1-\alpha)$, to spawn a new `0`-region or to disappear and it could also evolve with probability $\alpha(1-\alpha)^2$ to `10001` (a pattern where the spawning phenomenon is no more possible). So considering only the number of regions yields a non negative expectation. We have to deal with two problems with the pattern `10011`: the evolution towards `10001` and the fact that the probability to increase or decrease by one the number of regions is the same. For the first one, we do not have an answer yet. So we consider α large enough so that this phenomenon is negligible. Now, we assume $\alpha > 0.9999$, the bound is not tight and could be improve by tuning further the constants.

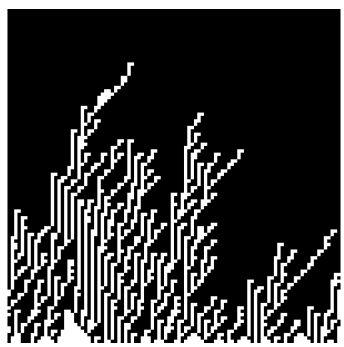
Our aim is to propose an answer to the second problem. Considering figure 3, one can notice that the `0`-regions are close to each other.



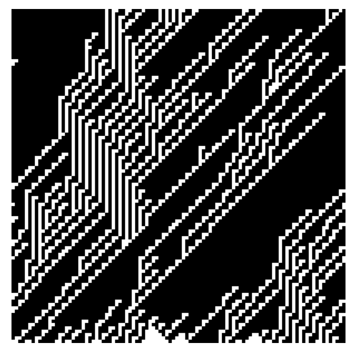
(a) $\alpha = 0.25$



(b) $\alpha = 0.5$



(c) $\alpha = 0.7$



(d) $\alpha = 0.9$

Fig. 2. BCF under different α-dynamics

Now considering the evolution of the pattern `10010` ($*$ means here `0` or `1`):

with probability	pattern $\mathtt{1}\underset{\times}{\mathtt{0}}\underset{\times}{\mathtt{0}}\mathtt{1}\underset{?}{\mathtt{0}}$ evolves to	distance to fixed point seems to
α^2	`1111`$*$	decrease
$\alpha(1-\alpha)$	`1101`$*$	slightly increase
$\alpha(1-\alpha)$	`1011`$*$	slightly increase
$(1-\alpha)^2$	`1001`$*$	slightly increase if $* =$ `1`

Because of the presence of a second `0`-region, no new `0`-region can spawn from the first one and the most likely evolution lead to the annihilation of the first `0`-region. When a `0`-region disappear in such a case, we will say that there is a collision between the two `0`-regions:

Definition 11. *We say that there is a* collision *when the first* `0`*-region in a pattern* `10010` *disappear because of an annihilation phenomenon.*

More importantly, if a pattern `10011` evolves to `10101`, the two `0`-regions are very close. So the probability that they collide does not seem to be negligible. If

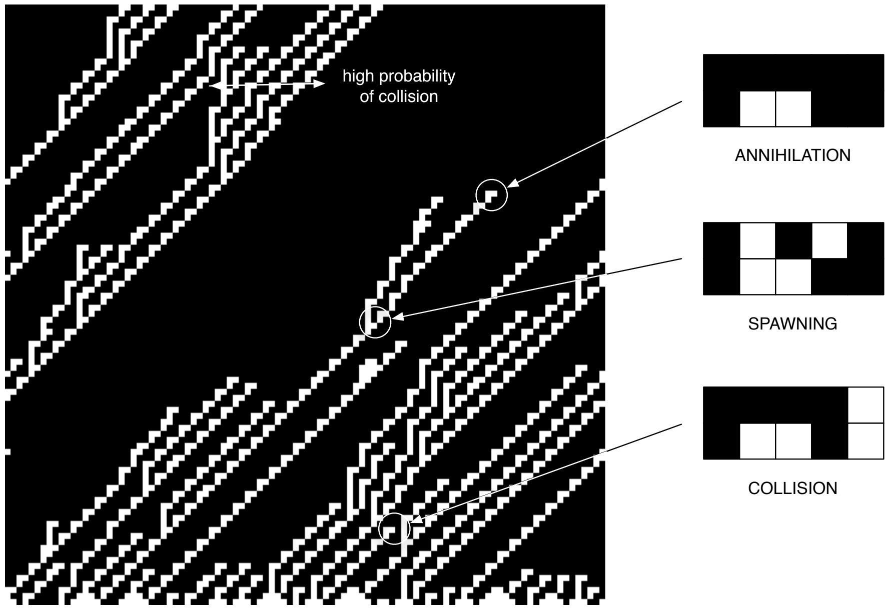


Fig. 3. explanation of the different phenomena

we can prove this than we can find local weights such that the variation of the local weights for a pattern `10011` is negative.

So we are interested in the evolution of a pattern `10101` and we would like to show that the probability that two `0`-regions collide is not negligible. We have chosen α so that the probability that at least two cells do not update in the considered patterns is negligible (as we later show in section 5). So we can consider that the pattern `10101` evolves almost under synchronous dynamics, and that sometimes one cell doesn't update. Figure 4 shows the evolution of a pattern `10101` when there is no collision with other `0`-regions. The black arrows show the most likely evolution (all cells update) and the dotted arrows show the evolution when one cell doesn't update. The weight of the leftmost `0`-region is written over it.

So, the most likely evolution of the pattern `10101` is to hit pattern `1011101`. After this, there will have a back and forth between this pattern and the pattern `10011001` until a cell does not update. Depending on which cell updates, it may evolve to a configuration where the probability of collision is too small to be considered or it may evolve to pattern `1011001` or `10011101`. In this case, it leads to a new back and forth between these two configurations until a cell does not update. Depending on which cell update, it may evolve to a configuration where the probability of collision is too small to be considered or it may evolve to pattern `100101` or `101101`. From these patterns, the most likely evolution leads

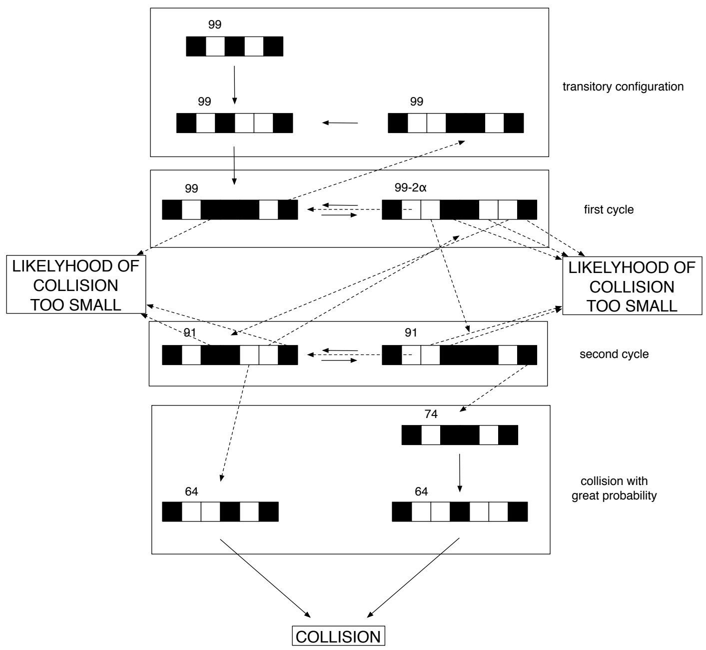


Fig. 4. evolution of the pattern `10101`

to a collision between the two `0`-regions. Now in the next section, we prove that the chosen weights define a function F which expected variation is negative at time step.

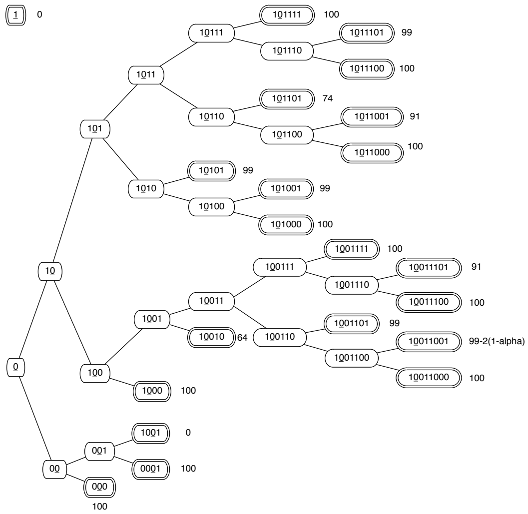
5 proof

5.1 Analysis

We now assume that $\alpha > 0.9999$. We consider the following variant. We use the masks basis and local weight function f given on figure 5. We have:

$$\begin{aligned}F(x) = {} & 64|x^t|_{\texttt{10010}} + 74|x^t|_{\texttt{101101}} + 91|x^t|_{\texttt{1011001}} + 91|x^t|_{\texttt{10011101}} \\ & +(99-2\alpha)|x^t|_{\texttt{10011001}} + 99|x^t|_{\texttt{1011101}} + 99|x^t|_{\texttt{10101}} + 99|x^t|_{\texttt{101001}} \\ & +99|x^t|_{\texttt{1001101}} + 100(|x^t|_{\texttt{101111}} + |x^t|_{\texttt{1011100}} + |x^t|_{\texttt{1011000}} + |x^t|_{\texttt{101000}} \\ & +|x^t|_{\texttt{1001111}} + |x^t|_{\texttt{10011100}} + |x^t|_{\texttt{10011000}} + |x^t|_{\texttt{1000}} + |x^t|_{\texttt{000}} + |x^t|_{\texttt{0001}}).\end{aligned}$$

For each configuration x, $F(x) \in \{0, 1, \ldots, 100n\}$ and $F(x) = 0$ if and only if $x = \texttt{1}^n$.



masks	$f(\dot{m})$
$10\dot{0}1$	0
$1\dot{0}010$	64
$1\dot{0}1101$	74
$1\dot{0}11001$, $1\dot{0}011101$	91
$1\dot{0}011001$	$99 - 2(1-\alpha)$
$1\dot{0}11101$, $1\dot{0}101$, $1\dot{0}1001$, $1\dot{0}01101$	99
autre	100

Fig. 5. Weight function for BCF.

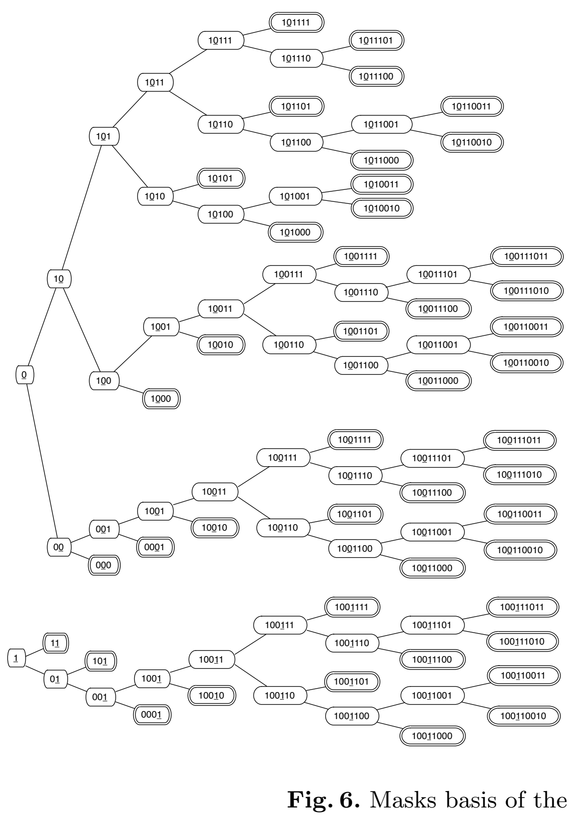


Fig. 6. Masks basis of the cases.

Lemma 2. *For all non-fixed point configuration x^t and configurations where there isn't only isolated* $\mathtt{0}$*,* $\mathbb{E}[\Delta F(x^t)] \leqslant -0.3(1-\alpha)$*. For configurations with only isolated* $\mathtt{0}$*,* $\mathbb{E}[\Delta F(x^t)] \leqslant 0$*.*

Proof. By linearity of expectation: $\mathbb{E}[\Delta F(x)] = \sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x,i)]$. We evaluate the variation of $F(x,i)$ using the masks basis of Figure 6.

Consider that at step t, cell i matches:

- **mask** $\underset{?}{\mathtt{1}}\dot{\mathtt{1}}$**,** $\mathtt{1}\underset{\times}{\mathtt{0}}\underset{\times}{\mathtt{0}}\dot{\mathtt{1}}\underset{?}{\mathtt{0}}$**:** $F(x^t,i) = 0$. With probability 1 at the step $t+1$, cell i matches mask $\dot{\mathtt{1}}$. So $F(x^{t+1},i) = 0$. Thus, $\Delta F(x^t,i) \leqslant 0$.
- **mask** $\underset{?}{\mathtt{0}}\dot{\mathtt{0}}\underset{?}{\mathtt{0}}$**,** $\mathtt{1}\dot{\mathtt{0}}\mathtt{1}\underset{\times}{\mathtt{0}}\mathtt{0}\underset{?}{\mathtt{0}}$**,** $\mathtt{1}\dot{\mathtt{0}}\underset{\times}{\mathtt{1}}\mathtt{1}\underset{\times}{\mathtt{0}}\mathtt{0}\underset{?}{\mathtt{0}}$**,** $\mathtt{1}\dot{\mathtt{0}}\underset{\times}{\mathtt{1}}\mathtt{1}\mathtt{1}\underset{\times}{\mathtt{0}}\underset{?}{\mathtt{0}}$**,** $\mathtt{1}\dot{\mathtt{0}}\underset{\times}{\mathtt{1}}\mathtt{1}\mathtt{1}\mathtt{1}$**:** $F(x^t,i) = 100$. Thus, $\mathbb{E}[(\Delta F(x^t,i)] \leqslant 0$.
- **mask** $\mathtt{1}\dot{\mathtt{0}}\underset{\times}{\mathtt{1}}\mathtt{1}\underset{\times}{\mathtt{0}}\underset{\times}{\mathtt{0}}\mathtt{1}\underset{?}{\mathtt{0}}$**:** $F(x^t,i) = 91$. Thus, $\mathbb{E}[(\Delta F(x^t,i)] \leqslant 9$.
- **mask** $\mathtt{1}\dot{\mathtt{0}}\mathtt{1}\underset{\times}{\mathtt{0}}\underset{\times}{\mathtt{0}}\mathtt{1}\underset{?}{\mathtt{0}}$**:** $F(x^t,i) = 99$. Thus, $\mathbb{E}[(\Delta F(x^t,i)] \leqslant 1$.
- **mask** $\mathtt{1}\mathtt{0}\underset{?}{\dot{\mathtt{1}}}$**:** $F(x^t,i) = 0$. With probability 1 at the step $t+1$, cell i matches mask $\dot{\mathtt{1}}$ or $\mathtt{1}\mathtt{0}\dot{\mathtt{0}}\mathtt{1}$. So $F(x^{t+1},i) = 0$. Thus, $\mathbb{E}[(\Delta F(x^t,i)] = 0$.
- **mask** $\mathtt{1}\dot{\mathtt{0}}\mathtt{1}\mathtt{0}\underset{?}{\mathtt{1}}$**:** With probability 1 at the step $t+1$, cell i matches mask $\mathtt{1}\dot{\mathtt{0}}\mathtt{1}\mathtt{0}\mathtt{1}$ or $\mathtt{1}\dot{\mathtt{0}}\mathtt{1}\mathtt{0}\mathtt{0}\mathtt{1}$. Thus, $\mathbb{E}[(\Delta F(x^t,i)] = 0$.
- **mask** $\mathtt{1}\underset{\times}{\dot{\mathtt{0}}}\mathtt{0}\underset{?}{\mathtt{0}}$**:** $F(x^t,i) = 100$. With probability α at the step $t+1$, cell i matches mask $\dot{\mathtt{1}}$ and $F(x^{t+1},i) = 0$. Thus, $\mathbb{E}[(\Delta F(x^t,i)] \leqslant -100\alpha \leqslant -99$.
- **mask** $\underset{?}{\mathtt{0}}\mathtt{0}\underset{\times}{\dot{\mathtt{0}}}\underset{?}{\mathtt{1}}$ **(and** $\underset{?}{\mathtt{0}}\mathtt{0}\underset{\times}{\mathtt{0}}\underset{?}{\dot{\mathtt{1}}}$ **together):**

With probability	At the step $t+1$, cell i matches mask	and $\Delta F(x^t,i)$	and $\Delta F(x^t,i+1)$
$\geqslant (1-\alpha)^2$	$\dot{\mathtt{0}}\mathtt{1}$	$\leqslant 0$	$= 0$
$\geqslant \alpha(1-\alpha)$	$\dot{\mathtt{1}}\mathtt{1}$	$= -100$	$= 0$
$\leqslant \alpha(1-\alpha)$	$\dot{\mathtt{0}}\mathtt{0}$	$\leqslant 0$	$\leqslant 100$
$\leqslant \alpha^2$	$\dot{\mathtt{1}}\mathtt{0}$	$= -100$	$\leqslant 100$

  (The two last cases are possible only if the state of cell $i+2$ is $\mathtt{1}$.)
  Thus, $\mathbb{E}[\Delta F(x^t,i) + \Delta F(x^t,i+1)] \leqslant 0$.
- **mask** $\mathtt{1}\underset{\times}{\dot{\mathtt{0}}}\underset{\times}{\mathtt{0}}\mathtt{1}\underset{?}{\mathtt{0}}$ **(and** $\mathtt{1}\underset{\times}{\mathtt{0}}\underset{\times}{\dot{\mathtt{0}}}\mathtt{1}\underset{?}{\mathtt{0}}$ **together):**

With probability	At the step $t+1$, cell i matches mask	and $\Delta F(x^t,i) + \Delta F(x^t,i+1)$
α^2	$\dot{\mathtt{1}}\mathtt{1}$	$= -64$
$1-\alpha^2$	other	$\leqslant 36$

  Since $\alpha > 0.9999$, $\mathbb{E}[\Delta F(x^t,i) + \Delta F(x^t,i+1)] \leqslant -64\alpha^2 + 36(1-\alpha^2) \leqslant -50$.
- **mask** $\mathtt{1}\dot{\mathtt{0}}\underset{\times}{\mathtt{1}}\mathtt{1}\mathtt{0}\underset{?}{\mathtt{1}}$**:**

With probability	At the step $t+1$, cell i matches mask	and $\Delta F(x^t,i)$
$= \alpha$	$\mathtt{1}\dot{\mathtt{0}}\mathtt{0}\mathtt{1}\mathtt{0}$	$= -10$
$\leqslant \alpha(1-\alpha)$	$\mathtt{1}\dot{\mathtt{0}}\mathtt{1}\mathtt{1}\mathtt{0}\mathtt{0}\mathtt{1}$	$= 17$
$\geqslant 1-\alpha(2-\alpha)$	$\mathtt{1}\dot{\mathtt{0}}\mathtt{1}\mathtt{1}\mathtt{0}\mathtt{1}$	$= 0$

  (The secund case is possible only if the state of cell $i+5$ is $\mathtt{1}$.)
  Since $\alpha > 0.9999$, $\mathbb{E}[\Delta F(x^t,i)] \leqslant -10\alpha + 17\alpha(1-\alpha) \leqslant -2$.
- **mask** $\mathtt{1}\dot{\mathtt{0}}\underset{\times}{\mathtt{1}}\mathtt{1}\mathtt{1}\mathtt{0}\underset{?}{\mathtt{1}}$**:**

With probability	At the step $t+1$, cell i matches mask	and $\Delta F(x^t,i)$
$\geqslant (1-\alpha)^2$	$1\dot{0}11101$	$=0$
$\leqslant \alpha(1-\alpha)$	$1\dot{0}11100$	$=1$
$\geqslant \alpha(1-\alpha)$	$1\dot{0}01101$	$=0$
$\leqslant \alpha^2$	$1\dot{0}011001$	$=-2(1-\alpha)$

If the state of cell $i+6$ is $\mathtt{0}$, then the second and fourth cases are impossible so $\mathbb{E}[\Delta F(x^t,i)] = 0$. Otherwise, since $\alpha > 0.9999$, $\mathbb{E}[\Delta F(x^t,i)] = \alpha(1-\alpha) - 2\alpha^2(1-\alpha) \leqslant 0$.
Thus, $\mathbb{E}[\Delta F(x^t,i)] \leqslant 0$.

– **mask** $1\dot{0}1\underset{\times}{0}\underset{\times}{0}\underset{\times}{1}1$**:**

With probability	At the step $t+1$, cell i matches mask	and $\Delta F(x^t,i)$
α^3	$1\dot{0}11101$	$=0$
$\alpha^2(1-\alpha)$	$1\dot{0}10101$	$=0$
$\alpha^2(1-\alpha)$	$1\dot{0}11111$	$=1$
$\alpha^2(1-\alpha)$	$1\dot{0}110011$	$=-8$
$(1-\alpha)^2(1+2\alpha)$	autre	$\leqslant 1$

Since $\alpha > 0.9999$, $\mathbb{E}[\Delta F(x^t,i)] \leqslant -7\alpha^2(1-\alpha)+(1-\alpha)^2(1+2\alpha) \leqslant -5(1-\alpha)$.

– **mask** $1\dot{0}\underset{\times}{1}1\underset{\times}{0}\underset{\times}{0}\underset{\times}{1}1$**:**

With probability	At the step $t+1$, cell i matches mask	and $\Delta F(x^t,i)$
α^4	$1\dot{0}011101$	$=0$
$\alpha^3(1-\alpha)$	$1\dot{0}010101$	$=-27$
$\alpha^3(1-\alpha)$	$1\dot{0}011111$	$=9$
$\alpha^3(1-\alpha)$	$1\dot{0}011001$	$\leqslant 8$
$\alpha^3(1-\alpha)$	$1\dot{0}111101$	$=9$
$(1-\alpha)^2(1+2\alpha+3\alpha^2)$	autre	$\leqslant 9$

Since $\alpha > 0.9999$, then $\mathbb{E}[\Delta F(x^t,i)] \leqslant -\alpha^3(1-\alpha)+9(1-\alpha)^2(1+2\alpha+3\alpha^2) \leqslant -0.8(1-\alpha)$.

– **mask** $1\underset{\times}{\dot{0}}\underset{\times}{0}\underset{\times}{1}111$ **(and** $1\underset{\times}{0}\underset{\times}{\dot{0}}\underset{\times}{1}111$**,** $1\underset{\times}{0}\underset{\times}{0}\underset{\times}{\dot{1}}111$ **together):**

With probability	At the step $t+1$, cell i matches mask	and $\Delta F(x^t,i)$	and $\Delta F(x^t,i+1)$	and $\Delta F(x^t,i+2)$
α^3	$1\dot{1}10111$	$=-100$	$=0$	$\leqslant 100$
$\alpha^2(1-\alpha)$	$1\dot{0}10111$	$=-1$	$=0$	$\leqslant 100$
$\alpha^2(1-\alpha)$	$1\dot{1}11111$	$=-100$	$=0$	$=0$
$\alpha^2(1-\alpha)$	$1\dot{1}001111$	$=-100$	$=100$	$=0$
$(1-\alpha)^2(1+2\alpha)$	other	$\leqslant 0$	$\leqslant 100$	$\leqslant 100$

Since $\alpha > 0.9999$, then $\mathbb{E}[\Delta F(x^t,i)+\Delta F(x^t,i+1)+\Delta F(x^t,i+2)] \leqslant -\alpha^2(1-\alpha) + 200(1-\alpha)^2(1+2\alpha) \leqslant (1-\alpha)(-\alpha^2+600(1-\alpha)) \leqslant -0.3(1-\alpha)$.

– **mask** $1\underset{\times}{\dot{0}}\underset{\times}{0}\underset{\times}{1}1\underset{\times}{0}0\underset{?}{0}$ **(and** $1\underset{\times}{0}\underset{\times}{\dot{0}}\underset{\times}{1}1\underset{\times}{0}0\underset{?}{0}$**,** $1\underset{\times}{0}\underset{\times}{0}\underset{\times}{\dot{1}}1\underset{\times}{0}0\underset{?}{0}$ **together):**

With probability	At the step $t+1$, cell i matches mask	and $\Delta F(x^t,i)$	and $\Delta F(x^t,i+1)$	and $\Delta F(x^t,i+2)$
α^3	$1\dot{1}101$	$=-100$	$=0$	$\leqslant 100$
$\alpha^2(1-\alpha)$	$1\dot{0}101$	$=-1$	$=0$	$\leqslant 100$
$\alpha^2(1-\alpha)$	$1\dot{1}111$	$=-100$	$=0$	$=0$
$\alpha^2(1-\alpha)$	$1\dot{1}001$	$=-100$	$\leqslant 100$	$=0$
$(1-\alpha)^2(1+2\alpha)$	other	$\leqslant 0$	$\leqslant 100$	$\leqslant 100$

As above, $\mathbb{E}[\Delta F(x^t,i)+\Delta F(x^t,i+1)+\Delta F(x^t,i+2)] \leqslant -0.3(1-\alpha)$.

- **mask** $1\underset{\times}{\dot{0}}\underset{\times}{0}\underset{\times}{1}10\underset{?}{1}$**(and** $1\underset{\times}{0}\underset{\times}{\dot{0}}\underset{\times}{1}10\underset{?}{1}$**,** $1\underset{\times}{0}\underset{\times}{0}\underset{\times}{\dot{1}}10\underset{?}{1}$ **together):**

With probability	At the step $t+1$, cell i matches mask	and $\Delta F(x^t,i)$	and $\Delta F(x^t,i+1)$	and $\Delta F(x^t,i+2)$
α^3	$1\dot{1}1010$	$=-99$	$=0$	$=99$
$\alpha^2(1-\alpha)$	$1\dot{0}1010$	$=0$	$=0$	$=99$
$\alpha^2(1-\alpha)$	$1\dot{1}111$	$=-99$	$=0$	$=0$
$\alpha^2(1-\alpha)$	$1\dot{1}0010$	$=-99$	$=64$	$=0$
$(1-\alpha)^2(1+2\alpha)$	other	$\leqslant 1$	$\leqslant 100$	$\leqslant 100$

Since $\alpha > 0.9999$, then $\mathbb{E}[\Delta F(x^t,i)+\Delta F(x^t,i+1)+\Delta F(x^t,i+2)] \leqslant -35\alpha^2(1-\alpha)+201(1-\alpha)^2(1+2\alpha) \leqslant (1-\alpha)(-35\alpha^2+603(1-\alpha)) \leqslant -15(1-\alpha)$.

- **mask** $1\underset{\times}{\dot{0}}\underset{\times}{0}\underset{\times}{1}11\underset{\times}{0}\underset{?}{0}$**(and** $1\underset{\times}{0}\underset{\times}{\dot{0}}\underset{\times}{1}11\underset{\times}{0}\underset{?}{0}$**,** $1\underset{\times}{0}\underset{\times}{0}\underset{\times}{\dot{1}}11\underset{\times}{0}\underset{?}{0}$ **together):**

With probability	At the step $t+1$, cell i matches mask	and $\Delta F(x^t,i)$	and $\Delta F(x^t,i+1)$	and $\Delta F(x^t,i+2)$
α^3	$1\dot{1}101$	$=-100$	$=0$	$\leqslant 100$
$\alpha^2(1-\alpha)$	$1\dot{0}101$	$=-1$	$=0$	$\leqslant 100$
$\alpha^2(1-\alpha)$	$1\dot{1}111$	$=-100$	$=0$	$=0$
$\alpha^2(1-\alpha)$	$1\dot{1}001$	$=-100$	$\leqslant 100$	$=0$
$(1-\alpha)^2(1+2\alpha)$	other	$\leqslant 0$	$\leqslant 100$	$\leqslant 100$

Since $\alpha > 0.9999$, then $\mathbb{E}[\Delta F(x^t,i)+\Delta F(x^t,i+1)+\Delta F(x^t,i+2)] \leqslant -\alpha^2(1-\alpha)+200(1-\alpha)^2(1+2\alpha) \leqslant (1-\alpha)(-\alpha^2+600(1-\alpha)) \leqslant -0.3(1-\alpha)$.

- **mask** $1\underset{\times}{\dot{0}}\underset{\times}{0}\underset{\times}{1}110\underset{\times}{1}1$**(and** $1\underset{\times}{0}\underset{\times}{\dot{0}}\underset{\times}{1}110\underset{\times}{1}1$**,** $1\underset{\times}{0}\underset{\times}{0}\underset{\times}{\dot{1}}110\underset{\times}{1}1$ **together):**

With probability	At the step $t+1$, cell i matches mask	$\Delta F(x^t,i)$	$\Delta F(x^t,i+1)$	$\Delta F(x^t,i+2)$
α^4	$1\dot{1}1011001$	$=-91$	$=0$	$=91$
$\alpha^3(1-\alpha)$	$1\dot{0}1011001$	$=8$	$=0$	$=91$
$\alpha^3(1-\alpha)$	$1\dot{1}1111001$	$=-91$	$=0$	$=0$
$\alpha^3(1-\alpha)$	$1\dot{1}0011001$	$=-91$	$\leqslant 99$	$=0$
$\alpha^3(1-\alpha)$	$1\dot{1}101101$	$=-91$	$=0$	$=74$
$(1-\alpha)^2(1+2\alpha+3\alpha^2)$	other	$\leqslant 9$	$\leqslant 100$	$\leqslant 100$

Since $\alpha > 0.9999$, then $\mathbb{E}[\Delta F(x^t,i)+\Delta F(x^t,i+1)+\Delta F(x^t,i+2)] \leqslant -\alpha^2(1-\alpha)+209(1-\alpha)^2(1+2\alpha+3\alpha^2) \leqslant (1-\alpha)(-\alpha^2+1254(1-\alpha)) \leqslant -0.8(1-\alpha)$.

- **mask** $1\underset{\times}{\dot{0}}\underset{\times}{0}\underset{\times}{1}1101\underset{?}{0}$**(and** $1\underset{\times}{0}\underset{\times}{\dot{0}}\underset{\times}{1}1101\underset{?}{0}$**,** $1\underset{\times}{0}\underset{\times}{0}\underset{\times}{\dot{1}}1101\underset{?}{0}$ **together):**

With probability	At the step $t+1$, cell i matches mask	and $\Delta F(x^t,i)$	and $\Delta F(x^t,i+1)$	and $\Delta F(x^t,i+2)$
α^3	$1\dot{1}101101$	$=-91$	$=0$	$=74$
$\alpha^2(1-\alpha)$	$1\dot{0}101101$	$=8$	$=0$	$=74$
$\alpha^2(1-\alpha)$	$1\dot{1}111$	$=-91$	$=0$	$=0$
$\alpha^2(1-\alpha)$	$1\dot{1}001101$	$=-91$	$=99$	$=0$
$(1-\alpha)^2(1+2\alpha)$	other	$\leqslant 9$	$\leqslant 100$	$\leqslant 100$

Since $\alpha > 0.9999$, then $\mathbb{E}[\Delta F(x^t,i)+\Delta F(x^t,i+1)+\Delta F(x^t,i+2)] \leqslant -17\alpha^3-\alpha^2(1-\alpha)+209(1-\alpha)^2(1+2\alpha) \leqslant -16$.

- **mask** $1\underset{\times}{\dot{0}}\underset{\times}{0}\underset{\times}{1}1\underset{\times}{0}\underset{\times}{0}\underset{\times}{1}1$**(and** $1\underset{\times}{0}\underset{\times}{\dot{0}}\underset{\times}{1}1\underset{\times}{0}\underset{\times}{0}\underset{\times}{1}1$**,** $1\underset{\times}{0}\underset{\times}{0}\underset{\times}{\dot{1}}1\underset{\times}{0}\underset{\times}{0}\underset{\times}{1}1$ **together):**

With probability	At the step $t+1$, cell i matches mask	and $\Delta F(x^t,i)$	$\Delta F(x^t,i+1)$	$\Delta F(x^t,i+2)$
α^6	$1\dot{1}1011101$	$=-99+2(1-\alpha)$	$=0$	$=99$
$\alpha^5(1-\alpha)$	$1\dot{0}1011101$	$\leqslant 1$	$=0$	$=99$
$\alpha^5(1-\alpha)$	$1\dot{1}1111101$	$\leqslant -98$	$=0$	$=0$
$\alpha^5(1-\alpha)$	$1\dot{1}0011101$	$\leqslant -98$	$=91$	$=0$
$\alpha^5(1-\alpha)$	$1\dot{1}1010101$	$\leqslant -98$	$=0$	$=99$
$\alpha^5(1-\alpha)$	$1\dot{1}1011111$	$\leqslant -98$	$=0$	$=100$
$\alpha^5(1-\alpha)$	$1\dot{1}1011001$	$\leqslant -98$	$=0$	$=91$
$(1-\alpha)^2(1+2\alpha+3\alpha^2+4\alpha^3+5\alpha^4)$	other	$\leqslant 2$	$\leqslant 100$	$\leqslant 100$

Since $\alpha > 0.9999$, then $\mathbb{E}[\Delta F(x^t, i)+\Delta F(x^t, i+1)+\Delta F(x^t, i+2)] \leqslant +2\alpha^6(1-\alpha) - 9\alpha^5(1-\alpha) + 202(1-\alpha)^2(1+2\alpha+3\alpha^2+4\alpha^3+5\alpha^4) \leqslant -5(1-\alpha)$.

– **mask** $\mathtt{1\underset{\times}{\dot{0}}\underset{\times}{0}\underset{\times}{1}1\underset{\times}{0}\underset{\times}{0}1\underset{?}{0}}$**(and** $\mathtt{1\underset{\times}{0}\underset{\times}{\dot{0}}\underset{\times}{1}1\underset{\times}{0}\underset{\times}{0}1\underset{?}{0}}$**,** $\mathtt{1\underset{\times}{0}\underset{\times}{0}\underset{\times}{\dot{1}}1\underset{\times}{0}\underset{\times}{0}1\underset{?}{0}}$ **together):**

With probability	At the step $t+1$, cell i matches mask	and $\Delta F(x^t, i)$	and $\Delta F(x^t, i+1)$	and $\Delta F(x^t, i+2)$
α^3	$\mathtt{1\dot{1}101}$	$\leqslant -98$	$= 0$	$\leqslant 100$
$\alpha^2(1-\alpha)$	$\mathtt{1\dot{0}101}$	$\leqslant 1$	$= 0$	$\leqslant 100$
$\alpha^2(1-\alpha)$	$\mathtt{1\dot{1}111}$	$\leqslant -98$	$= 0$	$= 0$
$\alpha^2(1-\alpha)$	$\mathtt{1\dot{1}001}$	$\leqslant -98$	$\leqslant 100$	$= 0$
$(1-\alpha)^2(1+2\alpha)$	other	$\leqslant 2$	$\leqslant 100$	$\leqslant 100$

Since $\alpha > 0.9999$, then $\mathbb{E}[\Delta F(x^t, i) + \Delta F(x^t, i+1) + \Delta F(x^t, i+2)] \leqslant 2\alpha^3 + 5\alpha^2(1-\alpha) + 202(1-\alpha)^2(1+2\alpha) \leqslant 3$.

Finally $\sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t, i)] \leqslant -99(|x^t|_{\mathtt{1000}}) - 50(|x^t|_{\mathtt{10010}}) + 9(|x^t|_{\mathtt{10110010}}) + (|x^t|_{\mathtt{1010010}}) + 3(|x^t|_{\mathtt{100110010}}) - 0.3(1-\alpha)(|x^t|_{\mathtt{100111}} + |x^t|_{\mathtt{1001101}} + |x^t|_{\mathtt{10011000}} + |x^t|_{\mathtt{100110011}}) \leqslant -0.3(1-\alpha)|x^t|_{\mathtt{100}}$. So, as long as x^t is not a fixed point or a configuration where all `0` are isolated, we have $\mathbb{E}[\Delta F(x^t)] \leqslant -0.3(1-\alpha)|x^t|_{\mathtt{100}} \leqslant -0.3(1-\alpha)$. And if x^t is a configuration where all `0` are isolated then $\mathbb{E}[\Delta F(x^t)] \leqslant 0$.

5.2 conclusion of the proof

Lemma 2 shows that F decrease on expectation for every configurations with at least a pattern `100`. But the variation of F is zero for configurations where all `0`s are isolated. Nevertheless if the configuration is not $(\mathtt{01})^{n/2}$ any modification done to these configurations leads to the creation of a pattern `1001`. If the configuration is $(\mathtt{01})^{n/2}$ then the automaton has hit a fixed point, we neglige this case in order to have an upper bound of the relaxation time. So if the configuration reach the set of the configurations where all `0`s are isolated which are not $(\mathtt{01})^{n/2}$, any change will get the configuration out of this set.

Now we consider the same automaton but we consider the sequence $(y^t)_{t>0}$ where $y^t = x^{2t}$ instead of $(x^t)_{t>0}$ (We only consider one configuration over two). Clearly the relaxation time in the later system is exactly twice as in the former system. We compute the expected variation of F in this case. We conclude that if y^t is a non fixed-point configuration where all `0`s are not isolated then $\mathbb{E}[\Delta F(y^t)] \leqslant -0.3(1-\alpha)$. Otherwise if y^t is a non fixed-point configuration where all `0`s are isolated, then with probability at least α at the time step $t+1$ there are a pattern `100` in x^{2t+1} and then $\mathbb{E}[\Delta F(y^t)] \leqslant -0.3\alpha(1-\alpha) \leqslant -0.2(1-\alpha)$.

Theorem 4. *If $\alpha > 0.9999$, any n-finite cyclic configuration under automaton **214** α-asynchronous dynamics converges a.s. to a fixed point. The relaxation time is $O\left(\frac{n}{(1-\alpha)}\right)$.*

Proof. Using Lemma 1 and Lemma 2, any n-finite cyclic configuration under automaton **214** α-asynchronous dynamics converges a.s. to a fixed point. The relaxation time is $O\left(\frac{n}{\alpha} \times \frac{1}{(1-\alpha)}\right) = O\left(\frac{n}{(1-\alpha)}\right)$.

6 conclusion

We have proven here that the relaxation time of automaton **214** under α-asynchronous dynamics is $O(\frac{n}{1-\alpha})$ when $\alpha > 0.9999$ while it diverges under both synchronous and totally asynchronous dynamics. Thus this automaton has a specific behavior under α-asynchronous dynamics. This result was obtained by exhibiting the collision phenomenon and neglecting the transitions where at most 2 cells does not update in the studied patterns. The bound we have found is not tight, it can probably be improved. Also automaton **210** is conjectured to have almost the same behavior. Slight modifications of the proof could probably yield similar results for automaton **210**. Automata **146** and **178** are the two remaining automata that have only been studied experimentally in [1]. They seem to exhibit a phase transition and as far as we know nothing has been done theoretically to prove these results.

From a technical point of view, this study also shows that mathematical analysis is possible. Shortening or automating the presented proof, and more generally designing mathematical tools to predict the behaviour in such probabilistic asynchronous dynamics, is an interesting issue.

Such prediction theorems and classifications would give precious information to modellers using cellular automata and would stand as valuable tools in addition to simulation

Note: the simulator used for the experiments is [10].

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References

1. Fatès, N., Regnault, D., Schabanel, N., Thierry, E.: Asynchronous behaviour of double-quiescent elementary cellular automata. In: LNCS 3887 Proc. of the 7th LATIN. (2006)
2. Fatès, N., Morvan, M., Schabanel, N., Thierry, E.: Asynchronous behaviour of double-quiescent elementary cellular automata. In: LNCS 3618 Proc. of the 30th MFCS. (2005)
3. Bersini, H., Detours, V.: Asynchrony induces stability in cellular automata based models. In: Proc. of the 4th Artificial Life. (1994) 382–387
4. Fatès, N., Morvan, M.: An experimental study of robustness to asynchronism for elementary cellular automata. Submitted, `arxiv:nlin.CG/0402016` (2004)
5. Fukś, H.: Probabilistic cellular automata with conserved quantities. arxiv:nlin.CG/0305051 (2005)
6. Fukś, H.: Non-deterministic density classification with diffusive probabilistic cellular automata. Phys. Rev. E **66** (2002)
7. Schönfisch, B., de Roos, A.: Synchronous and asynchronous updating in cellular automata. BioSystems **51** (1999) 123–143
8. Gács, P.: Deterministic computations whose history is independent of the order of asynchronous updating. http://arXiv.org/abs/cs/0101026 (2003)
9. Louis, P.Y.: Automates Cellulaires Probabilistes : mesures stationnaires, mesures de Gibbs associées et ergodicité. PhD thesis, Université de Lille I (2002)
10. Fatès, N.: Fiatlux CA simulator in Java. Sources and experimental data available from <http://perso.ens-lyon.fr/nazim.fates> (2004)