

On the analysis of "simple" 2D stochastic cellular automata

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Abstract. We analyze the dynamics of a two-dimensional cellular automaton, 2D Minority, for the Moore neighborhood (eight neighbors per cell) under fully asynchronous dynamics (where only one random cell updates at each time step). Even if 2D Minority seems a simple rule, from the experience of Ising models and Hopfield nets, it is known that models with negative feedback are hard to study. This automaton actually presents a rich variety of behaviors, even more complex than what was observed and analyzed in a previous work on 2D Minority for the von Neumann neighborhood (four neighbors per cell) [1], including particles and a wider range of stable configurations. Nevertheless our work suggests that predicting the behavior of this automaton although difficult is possible, opening the way to analyze the class of totalistic automata.

1 Introduction

Cellular automata are attractive models for complex systems in various fields, like physics, biology or social sciences. Their relevance is supported by many observations of natural phenomena which closely match the dynamics of some cellular automaton, as illustrated by Fig. 1. An example of challenging issue in biology is to predict the expression of genes in a set of cells which share the same gene regulatory network. Cellular automata can be used to model such systems [2, 3]. For example consider the simple gene regulatory networks where a gene exerts a feedback inhibition of its expression. The state of a cell is whether it expresses this gene or not. Assuming that each cell starts expressing the gene when less than half of its neighbors (including itself) express it, and that otherwise it stops expressing it, leads to the Minority rule [4]. If cells are assembled into a two-dimensional grid, it yields 2D Minority. Such a model is of course an extreme simplification of any real phenomena but understanding this "simple" model is already an indispensable step towards the study of more involved interaction networks. Surprisingly, it already exhibits an astonishingly rich behavior which is investigated in this paper.



1.a – The pattern growth of the shell of the widespread species *Conus textile* is governed by a mathematical function presenting similarities with the Rule 30 CA above. 1.b – 2D minority induced by a set of cells expressing (in black) or not (in white) a gene which tends to inhibit its expression in neighboring cells.

Fig. 1. Cellular automata as models in biology.

The 2D Minority automaton belongs to the class of threshold cellular automata which have been intensively studied under synchronous dynamics (at each time step, all the cells update simultaneously) [5]. However models for natural phenomena rather update asynchronously. Empirical studies [6–8] have shown how the behavior can change drastically when introducing asynchronism. However only few mathematical analyses are available and they mainly concern one-dimensional stochastic cellular automata [9–12]. Providing analyses of 2D rules remains a real challenge. For instance the mean-field approach does not succeed in approximating tightly such stochastic dynamics [13]. Fig. 2 illustrates for three 2D cellular automata the differences between the synchronous dynamics and the *fully asynchronous dynamics* where only one random cell updates at each time step. Some related stochastic models like Ising models or Hopfield nets have been studied under asynchronous dynamics (our model of asynchronism corresponds to the limit when temperature goes to 0 in the Ising model). These models are acknowledged to be harder to analyze when it comes to two-dimensional topologies [14] or negative feedbacks [15].

For all these reasons 2D Minority under fully asynchronous dynamics turns out to be an interesting and challenging candidate for mathematical analyses. This paper focuses on the Moore neighborhood: at each time step, the fired cell updates to the minority state among its eight closest neighbors and itself. It

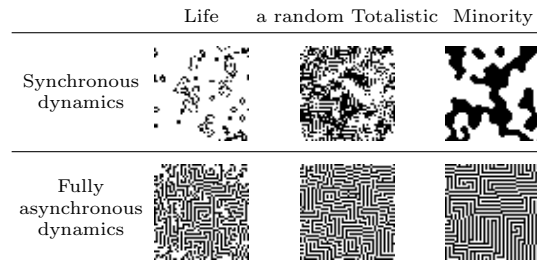


Fig. 2. Typical configurations observed during the evolution of some 2D cellular automata (Moore neighborhood). Similar stripes emerge in asynchronous regime even if their synchronous behavior differ drastically.

carries on a work initiated in [1] where 2D Minority was analyzed for the von Neumann neighborhood (four neighbors instead of eight). One might have hoped for minor adjustments to deal with the Moore neighborhood, however the results do not come out so easily. Experiments discussed in Section 3 show that new patterns (wider variety of striped patterns) and new phenomena occur (particle-like behaviors). Several key ideas presented in [1] (energy, borders and regions) apply, but their use requires some innovations. We show that the initial stage of the dynamics is characterized by a fast energy drop (Theorem 3). We exhibit borders that separate striped regions competing with one another and we manage to prove how final stable (horizontally or vertically) stripes configurations are reached almost surely from typical configurations occurring at the end of the process. Furthermore, we prove that this convergence occurs in polynomial time (Theorem 5). In the proof, we show that as the regions crumble, inflate or retract, the overall structure admits a recursive description which persists over time. The proofs of the study of such dynamical systems, known as complex systems, involve unavoidable tedious case studies and one of the important contributions of this paper is to set up a compact, possibly elegant, and thus safe framework to deal with these enumerations of cases. Note that in the course of the paper, we present an interesting characterization of the stable configurations for 2D Minority for the Moore neighborhood (Theorem 2). As far as we know, it had only been solved for the von Neumann neighborhood [5].

2 Definitions

We consider in this paper the 2D 2-states cellular automaton Minority under fully asynchronous dynamics over finite configurations with periodic boundary conditions. We are given two positive integers n and m , let $N = nm$. We denote by $\mathbb{T} = \mathbb{Z}_n \times \mathbb{Z}_m$ the set of *cells* and $Q = \{0, 1\}$ the set of *states* (0 stands for white and 1 for black in the figures). We consider the *Moore neighborhood*: two cells (i, j) and (k, l) are *neighbors* if $\max(|i - k|_n, |j - l|_m) \leq 1$ (where $|i - j|_p$ denotes the distance in \mathbb{Z}_p). A $n \times m$ -*configuration* c is a function $c : \mathbb{T} \rightarrow Q$; c_{ij} is the *state* of the cell (i, j) in configuration c .

We consider the *fully asynchronous dynamics of 2D Minority*. Time is discrete and let c^t denote the configuration at time t ; c^0 is the *initial configuration*. The configuration at time $t + 1$ is a random variable defined by the following process: a cell (i, j) is selected uniformly at random in \mathbb{T} and its state is updated to the minority state in its neighborhood (we say that cell (i, j) *fires* at time t), all the other cells remain in their current state: $c_{ij}^{t+1} = 1$ if $(c_{ij}^t + c_{i-1,j}^t + c_{i+1,j}^t + c_{i,j-1}^t + c_{i,j+1}^t + c_{i-1,j+1}^t + c_{i-1,j-1}^t + c_{i+1,j-1}^t + c_{i+1,j+1}^t) \leq 4$, and $c_{ij}^{t+1} = 0$ otherwise; and $c_{kl}^{t+1} = c_{kl}^t$ for all $(k, l) \neq (i, j)$. A cell is said *active* if its state would change if fired.

A configuration c is *stable* if it remains unchanged under the dynamics, *i.e.*, if all its cells are inactive. We say that the random sequence (c^t) *converges* almost surely from an *initial configuration* $c^0 = c$ if the random variable $T = \min\{t :$

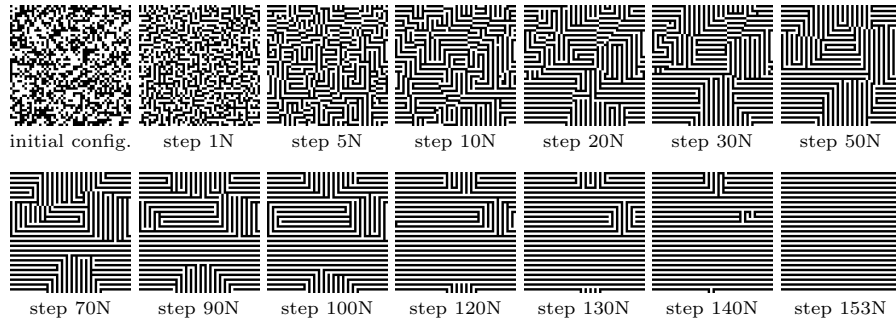





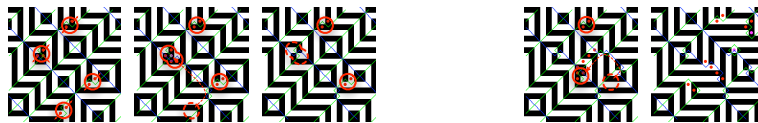
Fig. 3. A typical execution of stochastic 2D Moore minority with $N = 50 \times 50$ cells.

c^t is stable} is finite with probability 1. We say that the convergence occurs in *polynomial time* on expectation if $\mathbb{E}[T] \leq p(N)$ for some polynomial p .

3 Experiments

Typical behavior. Like other 2D automata (such as Game of Life [6]) the asynchronous behavior of 2D Minority differs radically from its synchronous dynamics. In particular, [5] proved that the synchronous dynamics eventually leads to stable configurations or cycles of two opposite configurations. The latter case is the typical behavior in synchronous simulations where one can observe big flashing islands (Fig. 2). On the contrary, as can be observed in Fig. 3, the states of most of the cells are very stable over time in fully asynchronous regime and present typically very rapidly striped patterns (horizontal or vertical) that tend to extend and merge with each other until one gets over the others and covers the whole configuration (when at least one of the dimensions n or m is even). A goal of this paper is to explore how such stripes arise and end up covering the whole configuration. Note also that such stripes arise as well in many other asynchronous automata such as the totalistic cellular automata (see Section 1). Very rarely a random initial configuration may converge to more exotic stable configurations. Fig. 6 gives some examples of more or less exotic stable configurations under 2D minority dynamics.

Borders and Particles. Part of the richness of 2D Minority under fully asynchronous behavior is due to some specific configurations where "particles" can be observed. Several patterns can be identified as particles although for now we do not have a formal definition. We say that there is a border between two diagonally neighboring cells if they are in the same state (more details in the next section). Active cells are always located near the borders. When the borders draw a pattern  (where red spots indicate active cells), then, depending of which of the two active cell fires the pattern will move in different directions: forward  or backward . Such patterns which "move" along borders can be called



4.a – A sequence of updates in a configuration starting with 4 particles where two of them move along rails and ultimately vanish after colliding with each other.

4.b – A sequence of updates where the rails cannot sustain the perturbations due to the movements of the particles: at some point, rails get too close with each other, new active cells appear, and part of the rail network collapses.

Fig. 4. Some examples of the complex behavior of particles in a 20×20 configuration.

particles. In some configurations, the set of all the borders form a network of "rails" carrying several particles. These particles follow random walks along the rails and vanish when they collide. Note that the dynamics is a lot more intricate than 2D random walks because the rails are modified by the passage of the particles and if two rails become too close, a whole part of the rail network collapses. Fig. 4 illustrates this kind of phenomena. Configurations with particles and rails are rarely reached from a random initial configuration. Nevertheless, we have to consider them when we study the convergence and these phenomena are extremely difficult to analyze mathematically. Such a system of particles is not observed in asynchronous 2D minority with von Neumann neighborhood [1] or in related models like the ferromagnetic Ising model or Hopfield networks with positive feedback.

4 Energy, Borders, Diamonds and Stable Configurations

4.1 Borders, Diamonds and Stripes

The following definitions allow to highlight the underlying structure of a configuration with respect to the dynamics. These tools turn out to be a key step to prove the convergence.

If n and m are even, a set of cells \mathcal{R} is said to be tiled with *even horizontal stripes* (resp. *odd horizontal*, *even* and *odd vertical stripes*) if $c_{ij} = i \bmod 2$ (resp. $i + 1$, j , $j + 1 \bmod 2$), for all cell $(i, j) \in \mathcal{R}$. Note that cells whose whole neighborhood is striped are inactive. We say that there is a *border* between two diagonally neighboring cells (i, j) and $(i + \epsilon, j + \eta)$, with $\epsilon, \eta \in \{-1, 1\}$, if they are in the same state, *i.e.*, if $c_{ij} = c_{i+\epsilon, j+\eta}$. If n and m are even, we say that a cell (i, j) is *even* (resp. *odd*) if $i + j$ is even (resp. odd). We say that the border between two cells is *blue* if the cells are even, and *green* otherwise; furthermore, we say that there is a *diamond* over cell (i, j) if its state coincides with even horizontal stripes, *i.e.*, if $c_{ij} = i \bmod 2$; the diamond is *blue* if the cell is even and *green* otherwise.

Proposition 1 (Borders are boundaries). *The borders are the exact boundary of regions tiled with stripes patterns (odd/even horizontal/vertical). More-*

over, when n and m are even, the blue (resp. green) borders are the exact boundary of the regions covered by the blue (resp. green) diamonds.

Since cells whose neighborhood is striped are inactive, the only active cells in a configuration may be found along the borders.

4.2 Energy

As in Ising model [14] or Hopfield networks [15], we define a natural global parameter that one can consider to be the energy of the system since it counts the number of interactions between neighboring cells in the same state. This parameter will provide key insights on the evolution of the system.

The *potential* v_{ij} of cell (i, j) is the number of its neighboring cells in the same state as itself minus 2.⁴ By definition, if $v_{ij} \leq 1$, then the cell is in the minority state in its own neighborhood and is thus *inactive* (its state will not change if fired); whereas, if $v_{ij} \geq 2$ then the cell is *active* and its state will change if fired. Note that a configuration c is *stable* iff for all cell $(i, j) \in \mathbb{T}$, $v_{ij} \leq 1$. Let say that a subset of cells \mathcal{R} is *fat* if for each cell $(i, j) \in \mathcal{R}$, there exists a square $Q = \{(i, j), (i + \epsilon, j), (i + \epsilon, j + \eta), (i, j + \eta)\}$, for some $\epsilon, \eta \in \{1, -1\}$, such that $Q \subset \mathcal{R}$. The *energy* $E_{\mathcal{R}}$ of set \mathcal{R} in a given configuration is defined as: $E_{\mathcal{R}} = \sum_{(i,j) \in \mathcal{R}} v_{ij}$. We denote by E the energy of the whole configuration c .

The next proposition shows that the energy is non-negative for almost every subset of cells of a configuration. This means that there cannot be too many cells with negative potential. This implies that the decrease of energy over time (Proposition 3 and Theorem 3) is not due to the increase of the number of cells with negative potential, but to the decrease of the potentials of the cells with positive potential, which explains intuitively why the striped patterns which have minimum energy (Proposition 4) arise naturally very rapidly.

Proposition 2 (Energy is non-negative). *For any fat subset of cells \mathcal{R} of size q : $0 \leq E_{\mathcal{R}} \leq 6q$.*

The following easy fact will be very handy in order to prove the convergence of the dynamics.

Fact 1. *When an active cell (i, j) is fired, its new potential is $v_{ij} := 4 - v_{ij}$ and the total energy varies by $8 - 2v_{ij}$. Note that if $v_{ij} = 2$, both remain unchanged.*

Proposition 3 (Energy is non-increasing). *Under fully asynchronous dynamics, the energy is a non-increasing function of time and decreases each time a cell with potential ≥ 3 fires.*

Proposition 4 (Minimum energy configurations). *The energy of a configuration c is 0 iff c is a striped configuration.*

⁴ The offset -2 is convenient since it ensures that the minimum energy of a configuration is 0 (see Proposition 2 below).

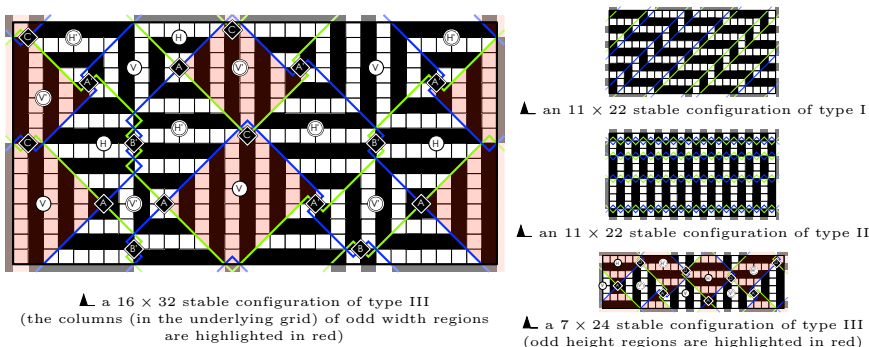




Fig. 5. Examples of stable configurations illustrating most of the possibilities.

4.3 Stable Configurations

As opposed to the von Neumann fully asynchronous dynamics in [1], stable configurations under the Moore neighborhood exhibit rather complex structures as shown on Fig. 6. Although there is a great variety of stable configurations, a general structure can be extracted and they can be characterized thanks to the borders. We first describe the stable configurations when n and m are even and deduce from there the structure of the stable configurations in the general case by doubling the odd dimension. Fig. 5 gives examples of each type of stable configurations.

Theorem 2 (Stable configurations). *When at least one of n and m is even, there are exactly three types of stable configurations:*

- *Type I: the borders are parallel straight (diagonal) lines such that: two lines of the same color are at (ℓ_1 -)distance ≥ 2 ; if two lines blue and green are at distance 1, there is no other line at distance ≤ 4 from each of them; the number of lines of each color along each row (resp., column) of the configuration has the same parity as m (resp., n).*
- *Type II: all the blue and green borders are all pairwise interlaced either horizontally according to the pattern , or vertically according to ; the pairs of interlaced borders are at distance ≥ 2 from each other; and the number of interlaced pairs has the parity of n if interlaced horizontally, and of m otherwise.*
- *Type III: the borders define a bicolor (horizontal/vertical stripes) underlying toric grid s.t.:*
 - *the segments of borders between two intersections are straight lines at distance at least 2 from each other;*
 - *two borders of the same color cannot intersect;*
 - *the number of borders of each color crossed by every row (resp. column) in the configuration has the same parity as m (resp. n);*

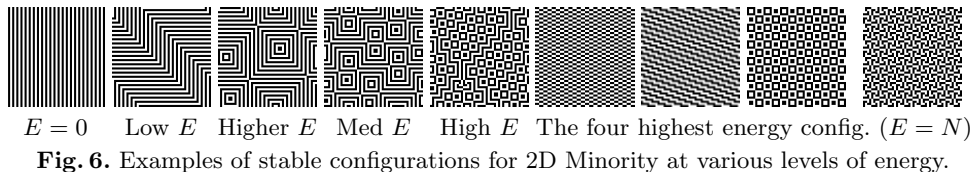
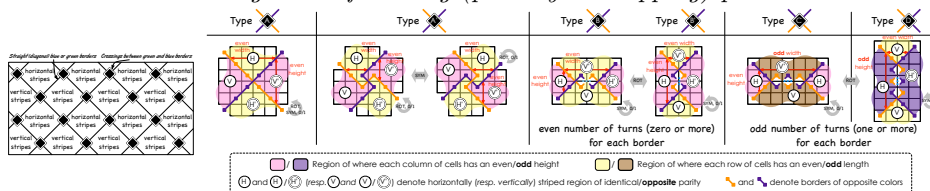


Fig. 6. Examples of stable configurations for 2D Minority at various levels of energy.

- the borders of opposite colors intersect at the corners of the cells only, and according to the following (possibly overlapping) patterns:



Furthermore, no stable configuration can have both crossings of types C and D and if a region has a crossing of type C (resp., D), all the crossings at the same vertical (resp., horizontal) level in the underlying grid are of type C (resp., D); moreover, the parity of the number of such horizontal (resp., vertical) levels of C-crossings (resp., D-) equals the parity of m (resp. n).

And any $n \times m$ -configuration of type I, II or III is a stable configuration.

Corollary 1. If n and m are odd, no stable configuration exists, and the dynamics never converges. If only one of n and m is odd, stable configurations of type I, II, and III exist with the parity restrictions mentioned in Theorem 2.

Proposition 5 (Stable configurations energy). The energy of a stable configuration c satisfies: $0 \leq E \leq N$. The only configurations with minimum energy (zero) are tiled with a striped 1×2 -pattern. And the only stable configuration with maximum energy N are of four types: either tiled with the 2×4 -pattern or the 8×8 -patterns , , or (see Fig. 6).

5 Analysis of the convergence

In this section, we give our results on the existence and speed of the convergence of the dynamics towards a stable configuration from an arbitrary initial configuration. As opposed to the von Neuman dynamics where we were able to analyse the whole convergence, because of the existence of particles following sophisticated guided random walks (see Section 3), we are only able to describe the first steps and the last steps of the convergence. These results rely on the study of the energy function which is combined with an other parameter to obtain a variant. This variant allows to reduce the study of the randomly evolving 2D shape to an one dimensional random walk. The section ends with challenging conjectures on the overall convergence of the process.

5.1 Initial energy drop

According to experiments, the energy of a configuration drops very fast during the first steps until it converges, most of the time to a striped configuration of minimal energy. The following theorem provides a bound on the speed of this initial energy drop.

Theorem 3. *The energy of any configuration of size N is at most $N + 2N/3$ after $O(N^2)$ fully asynchronous minority updates on expectation.*

5.2 The last steps of convergence

From now on, we assume that n and m are even. As mentioned above, in most of the experiments striped regions arise quickly, then they extend, compete with each other, merge until only one covers the whole configuration. In this section, we provide an analysis of the very last steps of the convergence to this stable configuration: the case where there remains only one *single horizontally striped region* within a *vertically striped background*, which we will call a standard configuration. We then show that the background ends up covering the whole configuration in polynomial time on expectation as expected according to the experiments (Fig. 3 when $t \geq 100N$). This involves studying the randomly evolving shape defined by the horizontally striped regions.

Note that every configuration is completely determined by its set of diamonds. When starting from a standard configuration, we show that the configuration exhibits a recursive structure that is conserved over time. We then show by studying a combination of the energy with the area of the random shape, that the random shape of the set of diamonds tends to vanish. Interestingly enough, we show that the horizontally striped region can flip the parity of its stripes but cannot extend beyond its initial surrounding rectangle. Let us now start with some definitions.

A *blue rectangle* (resp. *green rectangle*) is a rectangle such that its sides are parallel to the diagonals and its corners are located at the centers of odd (resp. even) cells. A blue or green rectangle is *enclosing* a set of diamonds D if all the diamonds are contained in the rectangle, and it is *surrounding* D if it is the smallest enclosing rectangle of that color for D . We say that a configuration is *standard* if it consists in a finite set of diamonds of the same color forming a rectangle (i.e., a set of diamonds of the same color whose borders match its surrounding rectangle). Two diamonds are *neighbors* if they have a side in common (and are thus of the same color). A set D of diamonds is: *connected* if D is connected for the neighborhood relationship; *convex* if for all $\epsilon \in \{1, -1\}$ and for any pair of diamonds centered on cells (i, j) and $(i + k, j + \epsilon k)$ in D , the diamonds centered on cells $(i + \ell, j + \epsilon \ell)$ for $0 \leq \ell \leq k$ belong to D ; an *island* if it is connected and convex.

Definition 1 (Valid configurations). *A valid configuration (or valid diamonds set) is defined recursively by a tree structure of interlocked blue or green rectangles where each subtree describes the diamond set enclosed within the corresponding rectangle. Precisely:*

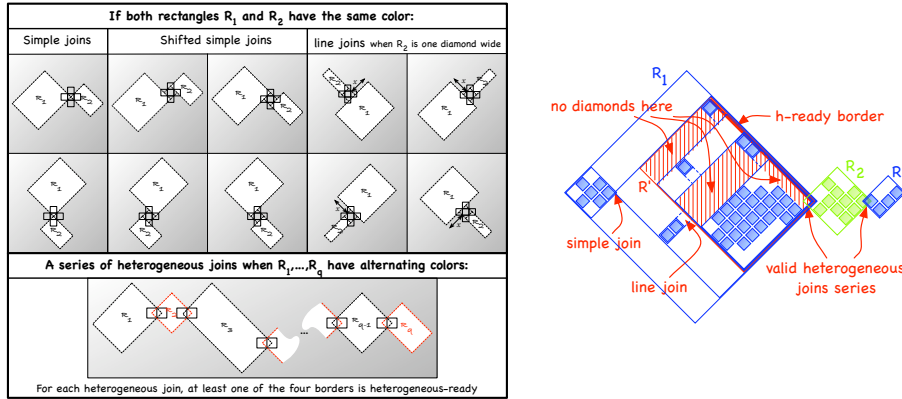


Fig. 7. To the left: valid combinations of valid configurations (the underlying cells of the automaton are shown at the junction of the rectangles). To the right: a valid configuration and its diamond set with a valid decomposition

- A set of diamonds consisting of an island is a valid configuration.
- The composition of two valid diamonds sets D_1 and D_2 enclosed by two rectangles R_1 and R_2 of the same color laying next to each other according to the patterns given in Fig. 7, is valid.
- The juxtaposition of q valid diamonds sets D_1, \dots, D_q enclosed in q rectangles R_1, \dots, R_q of alternating colors as shown in Fig. 7, is valid if at each junction, either both a blue and a green diamonds are located at the corresponding corners of the surrounding blue and green rectangles, or at least one of the four borders of these rectangles is h-ready; we say that the north-east border of an enclosing rectangle R of a valid configuration is h-ready if, within the smallest rectangle R' corresponding to the node enclosing all the diamonds laying along this border in the construction tree of R , the diamonds are located as follows: no diamond may lay in R' to the south-west of the diamonds along the north-east border of R nor one row below (see Fig. 7) (the definition extends naturally to NW, SW, and SE borders by rotation).

A configuration is valid if its corresponding set of diamonds is valid. Each valid configuration is recursively described by a construction tree: a binary tree where each leaf is an island and each internal node stands for a join operation whose two edges pointing downwards are labeled by the two, blue or green, joint rectangles enclosing the two valid diamond sets described by the left and right subtrees.

Fig. 7 gives an example of a valid configuration starting with a blue island composed with several line joins, followed by a simple join with another blue island, and ending with a series of two heterogeneous joins with the two islands to its right. A valid configuration can be represented by several construction trees. Rearranging construction trees according to certain rules is one of the keys to the following results (like Theorem 4).

The set of valid configurations is closed under the minority dynamics. In the fully asynchronous dynamics, only one cell fires at each time step, thus only one diamond is added or removed at each time step. Since there are horizontal stripes inside an island, the cells which are not at the borders are not active. All the deletions and additions of diamonds occur at the borders. A careful analysis of the actives cells in valid configuration yields the following theorem.

Theorem 4 (Closure and Reachability). *The set of valid configurations is the set of all reachable configurations from standard configurations.*

The energy function is not sufficiently precise to follow the evolution of the valid configurations since it may remain constant for long period of time whereas the configuration evolves towards a stable configuration. But combining it with the *area* A of the configuration, defined as the number of its diamonds, yields a variant from which we will deduce a polynomial bound on the expected convergence time.

Proposition 6. *The energy of a valid configuration is equal to twice the number of its blue and green borders minus twice the number of intersections of blue and green borders. Thus, $E \leq 8A$.*

The variant. Let $\Phi = A + E/4$, for any given configuration. Let us denote by $\mathbb{E}[\Delta\Phi]$ the expected variation of Φ for this configuration after one fully asynchronous minority update.

Proposition 7. *For any valid configuration constructed from k islands with ℓ joins: $\mathbb{E}[\Delta\Phi] \leq \frac{3\ell - 3k}{N}$.*

Proof. (Sketch) The proof proceeds by induction on the construction tree of the valid configuration. By following clockwise the borders of the island and counting the active cells, we can show that the expected variation of Φ for a configuration with only one island is at most $-\frac{3}{N}$. If the configuration is obtained by joining two valid configurations then whatever the join is, it can be checked that an active cell in one of the two configurations remains active with the same characterization in the joined configuration. A cell which is inactive in both configurations is inactive in the joined configuration, except around the join where at most three cells may have their activity changed. Then the expected variation of Φ is the sum of the expected variation of the two configurations plus the effect of these three cells which is in every case at most $+\frac{3}{N}$. \square

Theorem 5. *Every valid configuration of area A converges to the background configuration in finite time with probability 1. The expected convergence time is $O(AN)$, which is thus $O(N^2)$.*

Proof. The construction of a valid configuration can be expressed as a binary tree where the leaves are the islands and the internal nodes are the joins (an heterogenous series is encoded as a series of two-by-two joins). Thus, $\ell = k - 1$ if

the tree is not empty, and by Proposition 6 and 7, as long as the configuration is not stable, $\mathbb{E}[\Delta\Phi] \leq -\frac{3}{N}$. For the initial configuration $\Phi \leq 3A$ and the stable configuration with vertical stripes is the only configuration where $\Phi = 0$. Thus it converges in finite time with probability 1 to this stable configuration and the expected convergence time is $O(AN)$ (see e.g. Lemma 5 in [9]). \square

Conclusion. The behavior of 2D Minority with the Moore neighborhood under fully asynchronous dynamics is surprisingly rich and difficult to analyze. The approach outlined in [1] for the von Neumann neighborhood is useful. The analysis of the energy and of the competing regions requires however a very accurate comprehension of the combinatorics of the automaton, which turned out to be more complex for the Moore neighborhood. A key to complete the analysis seems to find the most appropriate definitions for particles and rails and explain precisely how they evolve.

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