

Progresses in the Analysis of Stochastic 2D Cellular Automata: a Study of Asynchronous 2D Minority

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Abstract. Cellular automata are often used to model systems in physics, social sciences, biology that are inherently asynchronous. Over the past 20 years, studies have demonstrated that the behavior of cellular automata drastically changed under asynchronous updates. Still, the few mathematical analyses of asynchronism focus on one-dimensional probabilistic cellular automata, either on single examples or on specific classes. As for other classic dynamical systems in physics, extending known methods from one- to two-dimensional systems is a long lasting challenging problem.

In this paper, we address the problem of analysing an apparently simple 2D asynchronous cellular automaton: 2D **Minority** where each cell, when fired, updates to the minority state of its neighborhood. Our experiments reveal that in spite of its simplicity, the minority rule exhibits a quite complex response to asynchronism. By focusing on the fully asynchronous regime, we are however able to describe completely the asymptotic behavior of this dynamics as long as the initial configuration satisfies some natural constraints. Besides these technical results, we have strong reasons to believe that our techniques relying on defining an energy function from the transition table of the automaton may be extended to the wider class of threshold automata.

Due to space constraint, we refer the reader to [16] for the missing proofs.

1 Introduction

In the literature, cellular automata have been both studied as a model of computation presenting massive parallelism, and used to model phenomena in physics, social sciences, biology... Cellular automata have been mainly studied under synchronous dynamics (at each time step, all the cells update simultaneously). But real systems rarely fulfill this assumption and the cell updates rather occur in an asynchronous mode often described by stochastic processes. Over the past 20 years, many empirical studies [2,4,5,13,18] have been carried out showing that the behavior of a cellular automaton may widely vary when introducing asynchronism, thus strengthening the need for theoretical framework to understand the influence of asynchronism. Still, the few mathematical analyses of the effects of asynchronism focus on one-dimensional probabilistic cellular automata, either on single examples like [8,9,15] or on specific classes like [6,7]. As for other

classic dynamical systems in physics, such as spin systems or lattice gas, extending known methods from one- to two-dimensional systems is a long lasting challenging problem. For example, understanding how a configuration all-up of spins within a down-oriented external field evolves to the stable configuration all-down has only recently been solved mathematically and only for the limit when the temperature goes to 0, *i.e.*, when only one transition can occur at time (see [3]). Similarly, the resolution of the study of one particular 2D automaton under a given asynchronism regime is already a challenge.

Our contribution. In this paper, we address the problem of understanding the asynchronous behavior of an apparently simple 2D stochastic cellular automaton: 2D **Minority** where each cell, when fired, updates to the minority state of its neighborhood. We show experimentally in Section 2 that in spite of its simplicity the minority rule exhibits a quite complex response to asynchronism. We are however able to show in Section 3 that this dynamics almost surely converges to a stable configuration (listed in Proposition 3) and that if the initial configuration satisfies some natural constraints, this convergence occurs in polynomial time (and thus is observable) when only one random cell is updated at a time. Our main result (Theorems 1 and 2) rely on extending the techniques based on one-dimensional random walks developed in [6,7] to the study of the two-dimensional random walks followed by the boundaries of the main components of the configurations under asynchronous updates. We have strong reasons to believe that our techniques relying on defining an energy function from the transition table of the automaton may be extended to the wider class of threshold automata.

Our results are of particular interest for modeling regulation network in biology. Indeed, 2D **Minority** cellular automaton represents an extreme simplification of a biological model where the biological cells are organized as a 2D grid and where the regulation network involves only two genes (the two states) which tend to inhibit each other (see [1]). The goal is thus to understand how the concentrations of each gene evolve over time within the biological cells, and in particular, which gene ends up dominating the other in each cell, *i.e.*, in which state ends up each cell. Understanding this simple rule is thus a key step in the understanding of more complex biological systems.

2 Experimental results

This section is voluntarily informal because it presents experimental observations whose formalizations are already challenging open questions. The next section will present in a proper theoretical framework our progresses in the understanding of these phenomena. The configurations studied here consist in a set of cells organized as a $n \times m$ torus (n and m are even) in which each cell can take two possible states: 0 (white) or 1 (black). The asynchronous behavior of 2D minority automaton turns out to be surprisingly complex for both of the studied neighborhoods:

- *von Neumann* (N-neighborhood for short), where each selected cell updates to the minority state within itself and its neighbors N, S, E, and W; and

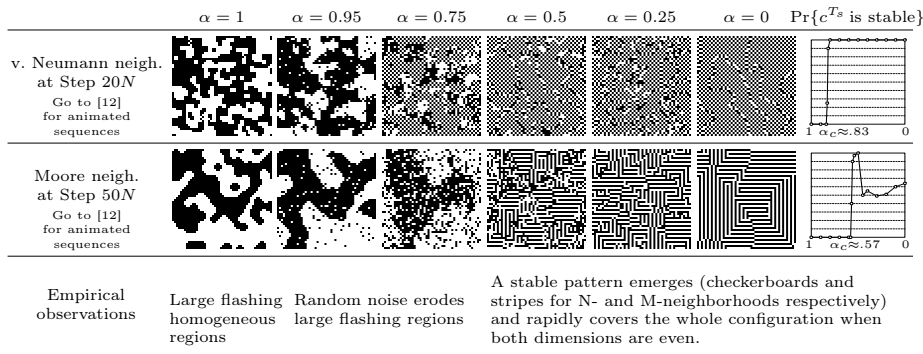


Fig. 1. 2D **Minority** under different α -asynchronous dynamics with $N_{50} = 50 \times 50$ cells. The last column gives, for $\alpha \in [0, 1]$, the empirical probability that an initial random configuration converges to a stable configuration before time step $T_s \cdot N_{50}$ where $T_s = 1000$ and $T_s = 2000$ for von Neumann and Moore neighborhood respectively.

- *Moore* (M-neighborhood for short), where each selected cell updates to the minority state among itself and its 8 closest neighbors N, S, E, W, NE, NW, SE, and SW.

In this section, we present a report on extensive experiments conducted on 2D **Minority** for both N- and M-neighborhood.

In this section, we consider the α -asynchronous 2D **Minority** dynamics in which at each time step, each cell updates to the minority state in its own neighborhood independently with probability α . We denote by $\alpha = 0$ the *fully asynchronous 2D Minority* dynamics in which at each time step, a daemon selects uniformly at random one cell and updates it to the minority state in its neighborhood.

The synchronous regime ($\alpha = 1$) of 2D **Minority** has been thoroughly studied in [10] where it is proved that it converges to cycles of length 1 or 2. Experimentally, from a random configuration, the synchronous dynamics in both neighborhoods converges to sets of large flashing white or black regions.

As soon as a little bit of asynchronism is introduced, the behavior changes drastically for both neighborhoods (see Fig. 1 and open our website [12] for animated sequences). Due to the asynchronism at each step, some random cells do not update and this creates a *noise* that progressively erodes the flashing homogenous large regions that were stable in the synchronous regime. After few steps, the configuration seems to converge rapidly to a homogeneous flashing background perturbed by random noise.

Experiments provide evidences that there exists a threshold α_c , $\alpha_c \approx .83$ and $\alpha_c \approx .57$ for the N- and M-neighborhoods respectively, such that if $\alpha \leq \alpha_c$, then stable patterns arise (*checkerboards* and *stripes* for N- and M-neighborhood respectively). As it may be observed in [12], above the threshold, when $\alpha > \alpha_c$, these patterns are unstable, but below and possibly at α_c , these patterns are sufficiently stable to extend and ultimately cover the whole configuration.

Convergence in asynchronous regimes. The last column of Fig. 1 shows that experimentally, when $\alpha \leq \alpha_c$, the asynchronous dynamics appears to converge at least with constant probability, rapidly to very particular stable configurations tiled by simple patterns known to be stable for the dynamics. Above the threshold, when $\alpha_c < \alpha < 1$, the asynchronous dynamics appears experimentally to be stuck into randomly evolving configurations in which no structure seems to emerge.

We will show in Theorem 1 that if at least one of the dimensions is even, the dynamics will almost surely reach a stable configuration, for all $0 \leq \alpha < 1$, but after at most an exponential number of steps. We conjecture that below the threshold α_c this convergence occurs in polynomial time on expectation if both dimensions are even (the threshold $T_s = 2000$ is probably too low for the M-neighborhood in Fig.1). We will prove this result in Theorem 2 for the fully asynchronous regime under the N-neighborhood under certain natural constraint on the initial configuration. Similar results to the ones to be presented below have been obtained in [17] for the M-neighborhood by extending of the techniques presented here.

3 Analysis of fully asynchronous 2D Minority

We consider now the fully asynchronous dynamics of 2D **Minority** with von Neumann neighborhood. Let n and m be two positive integers and $\mathbb{T} = \mathbb{Z}_n \times \mathbb{Z}_m$ the $n \times m$ -torus. A $n \times m$ -configuration c is a function $c : \mathbb{T} \rightarrow \{0, 1\}$ that assigns to each cell $(i, j) \in \mathbb{T}$ its state $c_{ij} \in \{0, 1\}$ (0 is white and 1 is black in the figures). We consider here the *von Neumann neighborhood*: the neighbors of each cell (i, j) are the four cells $(i \pm 1, j)$ and $(i, j \pm 1)$ (indices are computed modulo n and m , we thus consider periodic boundary conditions). We denote by $N = nm$, the total number of cells.

Definition 1 (Stochastic 2D Minority). *We consider the following dynamics δ that associates to each configuration c a random configuration c' obtained as follows: a cell $(i, j) \in \mathbb{T}$ is selected uniformly at random and its state is updated to the minority state in its neighborhood (we say that cell (i, j) is fired), all the other cells remain in their current state: $c'_{ij} = 1$ if $c_{ij} + c_{i-1,j} + c_{i+1,j} + c_{i,j-1} + c_{i,j+1} \leq 2$, and $c'_{ij} = 0$ otherwise; and $c'_{kl} = c_{kl}$ for all $(k, l) \neq (i, j)$. We say that a cell is active if its neighborhood is such that its state changes when the cell is fired.*

Definition 2 (Convergence). *We denote by c^t the random variable for the configuration obtained from a configuration c after t steps of the dynamics: $c^t = \delta^t(c)$; $c^0 = c$ is the initial configuration.*

We say that the dynamics δ converges almost surely from an initial configuration c^0 to a configuration \bar{c} if the random variable $T = \min\{t : c^t = \bar{c}\}$ is finite with probability 1. We say that the convergence occurs in polynomial (resp., linear, exponential) time on expectation if $\mathbb{E}[T] \leq p(N)$ for some polynomial (resp., linear, exponential) function p .







Von Neumann neighborhoods						
	Isolated	Peninsula	Corner	Bridge	Border	Surrounded
Minority $\delta(c)$	Inactive	Inactive	Active Reversible $\Delta E(c) = 0$	Active Reversible $\Delta E(c) = 0$	Active Irreversible $\Delta E(c) = -4$	Active Irreversible $\Delta E(c) = -8$
Outer-totalistic 976 $\hat{\delta}(\hat{c}) = \boxtimes \oplus \delta(\boxtimes \oplus \hat{c})$	Active Irreversible $\Delta E(c) = -8$	Active Irreversible $\Delta E(c) = -4$	Active Reversible $\Delta E(c) = 0$	Active Reversible $\Delta E(c) = 0$	Inactive	Inactive

Fig. 2. Neighborhood's names and transition tables of **Minority** δ and its counterpart Outer-Totalistic **976** $\hat{\delta}$ (see section 3.2): only active cells switch their states when fired.

As seen in Section 2, any configuration tend to converge under this dynamics towards a *stable configuration*, *i.e.*, towards a configuration where all cells are in the minority state of their neighborhood, *i.e.*, inactive.

Checkerboard patterns. We say that a subset of cells $R \subseteq \mathbb{T}$ is *connected* if R is connected for the neighborhood relationship. We say that R is *checkerboard-tiled* if all adjacent cells in R are in opposite states. A *horizontal* (resp., *vertical*) *band* of width w is a set of cells $R = \{(i, j) : k \leq i < k + w\}$ for some k (resp., $R = \{(i, j) : k \leq j < k + w\}$).

3.1 Energy of a configuration

The following natural parameters measure the stability of a configuration, *i.e.*, how far the cells of the configuration are from the minority state in their neighborhood. Following the seminal work of Tarjan in amortized analysis [19], we define a local potential that measures the amount of local unstability in the configuration. We proceed by analogy with the spin systems in statistical physics (Ising Model [3]): we assign to each cell a potential equal to the benefit of switching its state; this potential is naturally defined as the number of its adjacent cells to which it is opposed (*i.e.*, here, the number of cells which are in the same state as itself); summing the potentials over all the cells defines the total energy of the configuration at that time. As we consider arbitrary initial configuration, the system evolves *out-of-equilibrium* until it (possibly) reaches a stable configuration, thus its energy will vary over time; in particular, as will be seen in Proposition 1, its energy will strictly decrease each time an irreversible transition is performed (*i.e.*, each time a cell of potential ≥ 3 is fired). It turns out that this energy function plays a central role in defining, in Section 3.4, the variant that will be used to prove the convergence of the system. We will see in particular that as observed experimentally in Section 2, the system tends to reach configurations of minimal energy as one would expect in a real physical system.

Definition 3 (Energy). *The potential v_{ij} of cell (i, j) is the number of its four adjacent cells that are in the same state as itself. The energy of a configuration c is defined as the sum of the potentials of the cells: $E(c) = \sum_{i,j} v_{ij}$.*

Definition 4 (Borders and Homogeneous regions). *We say that there is a border between two neighboring cells if they are in the same state. An alternating*

path is a sequence of neighboring cells that does not go through a border, i.e., of alternating states. This defines an equivalence relationship «being connected by an alternating path», the equivalence classes of this relationship are called the homogenous regions of the configuration.

By definition, each homogeneous region is connected and tiled by one of the two checkerboard patterns, either $\begin{smallmatrix} \blacksquare & \square \\ \square & \blacksquare \end{smallmatrix}$ or $\begin{smallmatrix} \square & \blacksquare \\ \blacksquare & \square \end{smallmatrix}$. The boundary of each homogeneous region is exactly the set of borders touching its cells. Note that the potential of a cell is the number of borders among its sides. The energy of a configuration is thus twice the number of borders and a cell is active if and only if at least two of its sides are borders. It follows that: if both dimensions n and m have the same parity, $(\forall c) E(c) \in 4\mathbb{N}$; and $(\forall c) E(c) \in 2 + 4\mathbb{N}$ otherwise.

There are two configurations of maximum energy $4N$: *all-black* and *all-white*. If n and m are even, there are two configurations of energy zero: the two *checkerboards*. If n is even and m is odd, the minimum energy of a configuration is $2n$ and such a configuration consists in a checkerboard pattern wrapped around the odd dimension creating a vertical band of width 2 tiled with pattern $\begin{smallmatrix} \blacksquare & \square \\ \square & \blacksquare \end{smallmatrix}$.

Energy of stable configurations. A cell is inactive if and only if its potential is ≤ 1 . It follows that the energy of any stable configuration belongs to $\{0, 2, \dots, N\}$. Stable configurations are thus as expected of lower energy. If n and m are even and at least one of them is a multiple of 4, there are stable configurations of maximum energy N , tiled by the “fat”-checkerboard $\begin{smallmatrix} \blacksquare & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \blacksquare \end{smallmatrix}$ or $\begin{smallmatrix} \square & \square & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \square & \square \end{smallmatrix}$.

Under the fully asynchronous dynamics δ , the overall variation of the energy of the configuration when the state of a cell of potential v is flipped is $8 - 4v \leq 0$, and since active cells have potential ≥ 2 :

Proposition 1 (Energy is non-increasing). *From any initial configuration c , the random variables $E(c^t)$ form a non-increasing sequence and $E(c^t)$ decreases by at least 4 each time a cell of potential ≥ 3 is fired.*

Initial energy drop. Furthermore, after a polynomial number of steps and from any *arbitrary* initial configuration, the energy falls rapidly below $5N/3$, which is observed experimentally through the rapid emergence of checkerboard patterns in the very first steps of the evolution. Observing that for any configuration of energy at least $5N/3$, there exists a sequence of at most two updates that decreases strictly the energy, one can show that:

Proposition 2 (Initial energy drop, proof omitted). *The random variable $T = \min\{t : E(c^t) < 5N/3\}$ is almost surely finite and $\mathbb{E}[T] = O(N^2)$.*

Every inactive cell touches at most one border. Thus, the boundaries of homogeneous regions in a stable configuration form straight lines at least 2 cells apart from each other. Thus,

Proposition 3 (Stable configurations). *Stable configurations are the configurations composed of parallel checkerboard-tiled bands of width at least 2. In particular, if n and m are odd, no stable configuration exists.*

It follows that if n and m are odd, the dynamics δ never reaches a stable configuration.

3.2 Coupling with Outer-Totalistic 976

From now on up to the end of section 3, we assume that n and m are even (with the only exception of Corollary 1). We denote by \boxtimes the checkerboard configuration of energy 0 defined as follows: $\boxtimes_{ij} = (i + j) \bmod 2$. Given two configurations c and c' , we denote by $c \oplus c'$ the XOR configuration c'' such that $c''_{ij} = (c_{ij} + c'_{ij}) \bmod 2$.

Dual configurations. As observed above, the fully asynchronous dynamics c^t tends to converge from any initial configuration c^0 to configurations tiled by large checkerboard regions. It is thus convenient to consider instead, the sequence of *dual configurations* (\hat{c}^t) defined by $\hat{c}^t = \boxtimes \oplus c^t$, in which the large checkerboard regions of c^t appear as large homogeneous black or white regions. Clearly, the dual sequence \hat{c}^t evolves according to the dynamics $\hat{\delta}(\cdot) = \boxtimes \oplus \delta(\boxtimes \oplus \cdot)$, indeed for all t , $\hat{c}^{t+1} = \boxtimes \oplus c^{t+1} = \boxtimes \oplus \delta(c^t) = \boxtimes \oplus \delta(\boxtimes \oplus \hat{c}^t) = \hat{\delta}(\hat{c}^t)$.

By construction, the two dual random sequences (c^t) and (\hat{c}^t) as well as their corresponding dynamics δ and $\hat{\delta}$ are *coupled probabilistically* (see [14]): the *same* random cell is fired in both configurations at each time step. A simple calculation shows that the dual dynamics $\hat{\delta}$ associates to each dual configuration \hat{c} , a dual configuration \hat{c}' as follows: select uniformly at random a cell (i, j) (the same cell (i, j) as δ fires on the primal configuration c), let $\Sigma = \hat{c}_{i-1,j} + \hat{c}_{i+1,j} + \hat{c}_{i,j-1} + \hat{c}_{i,j+1}$ and set: $\hat{c}'_{ij} = 1$ if $\Sigma \geq 3$; $\hat{c}'_{ij} = 1 - \hat{c}_{ij}$ if $\Sigma = 2$; and $\hat{c}'_{ij} = 0$ otherwise; and $\hat{c}'_{kl} = \hat{c}_{kl}$ for all $(k, l) \neq (i, j)$. It turns out that this rule corresponds to the asynchronous dynamics of the cellular automaton Outer-Totalistic 976 [11]. The corresponding transitions are given in Fig. 2.

Stable configurations of Outer-Totalistic 976. We define the energy of the dual configuration \hat{c} and the potentials of each of its cells (i, j) as the corresponding quantities, $E(\hat{c})$ and v_{ij} , in the primal configuration c . By Proposition 3, the stable dual configurations under the dual dynamics $\hat{\delta}$ are the dual configurations composed of homogeneous black or white bands of widths ≥ 2 . The two dual configurations of minimum energy 0 are all-white and all-black.

Experimentally, any dual configuration under the fully asynchronous dynamics $\hat{\delta}$ evolves towards large homogeneous black or white regions (corresponding to the checkerboard patterns in the primal configuration). Informally, these regions evolve as follows (see Fig. 2): isolated points tend to disappear as well as peninsulas; borders and surrounded points are stable; large regions are eroded in a random manner from the corners or bridges that can be flipped reversibly and their boundaries follow some kind of 2D random walks until large bands without corners ultimately survive (see Fig. 3 or [12]).

3.3 Convergence from an arbitrary initial configuration

In this section, we consider *arbitrary* initial configurations c^0 and show that indeed the dynamics δ converges to a stable configuration almost surely and after at most an exponential number of steps on expectation.

Theorem 1. *From any initial configuration c^0 , the dynamics δ converges to a stable configuration after at most $2N^{2N+1}$ steps on expectation.*

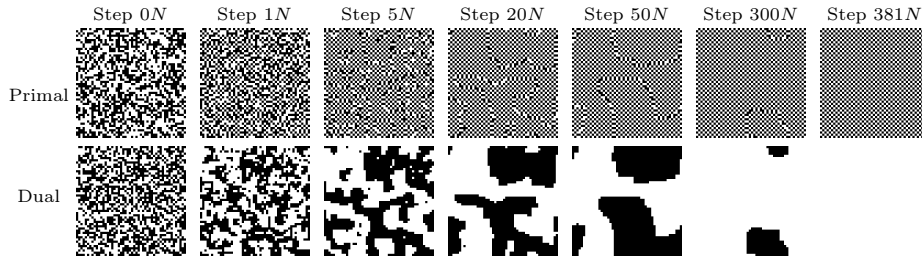


Fig. 3. The coupled evolutions of **Minority** δ on the primal configurations (c^t) (above) and its counterparts Outer-Totalistic **976** $\hat{\delta}$ on dual configurations (\hat{c}^t) (below). Note that from step $50N$ on, (c^t) an (\hat{c}^t) are bounded configurations.

Proof. According to the coupling above, it is equivalent to prove this statement for the dual dynamics. The following sequence of $\hat{\delta}$ -updates transforms any dual configuration \hat{c} into a dual stable configuration : *I*) as long as there are active white cells, choose one of them and switch its state to black; *II*) as long as there are active black cells, choose one of them and switch its state to white.

During phase 1, the black regions expand until they fill their surrounding bands or surrounding rectangles. Clearly according to the transition table Fig. 2, after phase 1 of the algorithm, every white cell is inactive and thus is either a border or surrounded. In particular, no white band of width 1 survived. During phase 2, the black cells enclosed in rectangles or in bands of width 1 are eroded progressively and ultimately disappear. Finally, only black bands of width ≥ 2 survive at the end of phase 2 and the configuration is stable since it is composed of homogeneous white or black bands of width ≥ 2 (see Proposition 3). During each phase, at most N cells change their state. We conclude that, from any configuration \hat{c} , there exists a path of length at most $2N$ to a stable configuration. Now, splits the sequence (c^t) into segments ($c^{2Nk+1}, \dots, c^{2N(k+1)}$) of length $2N$. The sequence of updates in each of these segments has a probability $1/N^{2N}$ to be the sequence of at most $2N$ updates given above that transforms configuration c^{2Nk} into a stable configuration. Since these events are independent, this occurs after N^{2N} trials on expectation. We conclude that the dynamics $\hat{\delta}$ and thus δ converge to a stable configuration after at most $2N \cdot N^{2N}$ steps on expectation. \square

Corollary 1. (*Proof omitted*) From any initial $n \times m$ -configuration c^0 , where n is even and m is odd, the dynamics δ converges to a stable configuration after at most $3N^{3N+1}$ steps on expectation.

3.4 Convergence from a bounded configuration

We consider again that n and m are even. We observe experimentally that most of the time, the dynamics converges rapidly to one of the two checkerboard configurations of energy zero. We demonstrate in this section that if the dynamics reaches a configuration composed of an arbitrary region surrounded by a checkerboard, then it will converge to the corresponding checkerboard configuration almost surely after a polynomial number of steps on expectation. This

corresponds to the analysis of the last steps of the behavior observed in experimentation. We believe that the techniques developed here may be extended to prove that the dynamics converges to a stable configuration in polynomial expected time from any initial configuration (see discussions in section 4).

Definition 5 (Bounded configuration). *We say that a configuration c is bounded if there exists a $(n-2) \times (m-2)$ rectangle such that the states in c of the cells outside this rectangle are equal to the corresponding states in one of the two checkerboard configurations. W.l.o.g., we assume that the upper-left corner of the rectangle is $(1, 1)$ and that the checkerboard is \blacklozenge , i.e., a configuration c is bounded if $c_{ij} = (i + j) \bmod 2$ for all $(i, j) \in \{(i, j) : (-1 \leq i \leq 0) \text{ or } (-1 \leq j \leq 0)\}$.*

Each cell outside the surrounding rectangle has 3 neighbors in an opposite state as itself, and is thus inactive. It follows that if c is a bounded configuration, $\delta(c)$ is also bounded within the same surrounding rectangle. A bounded configuration is thus equivalent to a finite perturbation of an infinite planar configuration in \mathbb{Z}^2 tiled with the \blacklozenge pattern. Since the dual of \blacklozenge is the configuration all-white, the dual of a bounded configuration is thus equivalent to a *finite number of black cells*, included into a $(n - 2) \times (m - 2)$ rectangle within an *infinite white planar configuration in \mathbb{Z}^2* . We shall now consider this setting.

Definition 6 (Convexity). *We say that a set of cells $R \subseteq \mathbb{Z}^2$ is convex if for any pair of cells (i, j) and $(i + k, j)$ (resp., $(i, j + k)$) in R , the cells $(i + \ell, j)$ (resp., $(i, j + \ell)$) for $0 \leq \ell \leq k$ belong to R . We say that R is an island if R is connected and convex.*

Our proof of the convergence of the dynamics in polynomial time for bounded configurations relies on the definition of a variant which decreases on expectation over time. It turns out that in order to define the variant, we do not need to consider the exact internal structure of the bounded configuration, but only the structure of the convex hull of its black cells.

Definition 7 (Convex hull of a configuration). *For any finite set of cells $R \in \mathbb{Z}^2$, we denote by $\text{hull}(R)$ the convex hull of the cells in R , i.e., $\text{hull}(R) = \bigcap \{S \subseteq \mathbb{Z}^2 : S \text{ is convex and } S \supseteq R\}$. Given a bounded dual configuration \hat{c} , we define the convex hull of \hat{c} , $\text{hull}(\hat{c})$, as the dual configuration whose black cells are the cells in the convex hull of the black cells of \hat{c} , i.e., if $R = \{(i, j) : \hat{c}_{ij} = 1\}$, $\text{hull}(\hat{c})_{ij} = 1$ if and only if $(i, j) \in \text{hull}(R)$. We say that a configuration c is convex if $\hat{c} = \text{hull}(\hat{c})$.*

We say that $\hat{c} \leq \hat{c}'$ if for all (i, j) , $c_{ij} \leq c'_{ij}$. Let \hat{c} be a convex dual bounded configuration. We define for each black cell (i, j) in \hat{c} , the island of \hat{c} that contains cell (i, j) , as the maximum connected and convex configuration \hat{c}' such that $\hat{c}'_{ij} = 1$ and $\hat{c}' \leq \hat{c}$. This defines a unique decomposition into black islands of the convex bounded configuration \hat{c} .

The variant. We now consider the following *variant*: $\Phi(\hat{c}) = E(\text{hull}(\hat{c}))/4 + |\text{hull}(\hat{c})|$, where $|\text{hull}(\hat{c})|$ is the number of black cells in the convex hull configuration $\text{hull}(\hat{c})$. We will show that from any initial configuration c^0 , $\Phi(c^t)$

decreases by at least $1/N$ on expectation at each time step until it reaches the value 0, *i.e.*, until the primal and dual configurations c^t and \hat{c}^t converge to the infinite checkerboard and the infinite all-white configurations respectively. In order to prove that $\Phi(c^t)$ decreases on expectation, we need to study the evolution of the convex hull of \hat{c}^t ; for this purpose, we introduce a modified coupled dual dynamics $\bar{\delta}$ that preserves the convexity of a dual configuration. Given a dual configuration \hat{c} , we denote by $\bar{\delta}(\hat{c})$ the random configuration \hat{c}' such that: $\hat{c}' = \bar{\delta}(\hat{c})$ if the cell updated by $\bar{\delta}$ is *not* a black bridge, and $\hat{c}' = \hat{c}$ otherwise. Since only firing a black bridge can break the convexity of a black region, then:

Lemma 1. *If \hat{c} is a convex bounded configuration, $\bar{\delta}(\hat{c})$ is a convex bounded configuration.*

The energy of a convex region is twice the number of borders, *i.e.*, twice the sum of the perimeters of the islands that compose it, so:

Lemma 2. *For all convex bounded configurations \hat{c} and \hat{c}' , if $\hat{c} \leq \hat{c}'$, then $E(\hat{c}) \leq E(\hat{c}')$.*

The construction of $\bar{\delta}$ guarantees that the image of the convex hull of \hat{c} by the dynamics $\bar{\delta}$ bounds from above the convex hull of the image of \hat{c} by the dynamics $\hat{\delta}$.

Lemma 3. *(Proof omitted) For all bounded configuration \hat{c} , $\hat{\delta}(\hat{c}) \leq \bar{\delta}(\text{hull}(\hat{c}))$.*

Let $\Delta\Phi_\lambda(\hat{c})$ be the random variable for the variation of the variant after one step of a dynamics λ from a configuration c , *i.e.*, $\Delta\Phi_\lambda(\hat{c}) = \Phi(\lambda(\hat{c})) - \Phi(\hat{c})$.

Corollary 2. *For all bounded configuration \hat{c} , $\Delta\Phi_{\bar{\delta}}(\hat{c}) \leq \Delta\Phi_{\bar{\delta}}(\text{hull}(\hat{c}))$.*

Proof. By definition, $\Delta\Phi_{\bar{\delta}}(\text{hull}(\hat{c})) - \Delta\Phi_{\bar{\delta}}(\hat{c}) = (|\bar{\delta}(\text{hull}(\hat{c}))| - |\text{hull}(\bar{\delta}(\hat{c}))|) + (E(\bar{\delta}(\text{hull}(\hat{c}))) - E(\text{hull}(\bar{\delta}(\hat{c}))))$. According to lemma 3, $\text{hull}(\bar{\delta}(\hat{c})) \leq \bar{\delta}(\text{hull}(\hat{c}))$ and thus $|\text{hull}(\bar{\delta}(\hat{c}))| \leq |\bar{\delta}(\text{hull}(\hat{c}))|$. And by Lemma 2, since both configurations are convex, $E(\text{hull}(\bar{\delta}(\hat{c}))) \leq E(\bar{\delta}(\text{hull}(\hat{c})))$. \square

Lemma 4. *For all bounded configuration \hat{c} that consists of a unique black island, $-4/N \leq \mathbb{E}[\Delta\Phi_{\bar{\delta}}(\hat{c})] \leq -3/N$.*

Proof. Each active cell is fired with probability $1/N$. According to the dynamics of $\bar{\delta}$ (the same as the dynamics of $\hat{\delta}$, Fig. 2, except that black bridges are inactive), if \hat{c} consists of an island of size ≥ 2 , $\mathbb{E}[\Delta\Phi_{\bar{\delta}}(\hat{c})] = -\frac{1}{N} (\#\{\text{black corners}\} + 2\#\{\text{black peninsulas}\}) + \frac{1}{N} \#\{\text{white corners}\} = -\frac{1}{N} \#\{\text{salient angles}\} + \frac{1}{N} \#\{\text{reflex angles}\} = -\frac{4}{N}$, since $\#\{\text{salient angles}\} - \#\{\text{reflex angles}\} = 4$ for all convex rectilinear polygon. Finally, if \hat{c} consists of a unique (isolated) black cell, $\Delta\Phi_{\bar{\delta}}(\hat{c}) = -3/N$. \square

Lemma 5. *For any bounded not-all-white configuration \hat{c} , $\mathbb{E}[\Delta\Phi_{\bar{\delta}}(\hat{c})] \leq -\ell/N$, where ℓ is the number of islands that compose $\text{hull}(\hat{c})$.*

Proof. By Corollary 2, $\mathbb{E}[\Delta\Phi_{\delta}(\hat{c})] \leq \mathbb{E}[\Delta\Phi_{\delta}(\text{hull}(\hat{c}))]$. By convexity of $\text{hull}(\hat{c})$, the sets of rows and columns touched by the islands that compose $\text{hull}(\hat{c})$ are pairwise disjoint. Thus, one can index the islands from 1 to ℓ from left to right, and the contacts between islands can only occur between two consecutive islands at the corners of their surrounding rectangles. Each contact creates at most two new active white cells that contribute for $+1/N$ each to $\mathbb{E}[\Delta\Phi_{\delta}(\text{hull}(\hat{c}))]$. The contribution of each island to $\mathbb{E}[\Delta\Phi_{\delta}(\text{hull}(\hat{c}))]$ is at most $-3/N$ according to Lemma 4. It follows that $\mathbb{E}[\Delta\Phi_{\delta}(\text{hull}(\hat{c}))] \leq -3\ell/N + 2(\ell - 1)/N \leq -\ell/N$. \square

Theorem 2. *The fully asynchronous minority dynamics δ converges almost surely from any initial bounded configuration c to the stable configuration of minimum energy, \blacksquare , and the expected convergence time is $O(AN)$ where A is the area of surrounding rectangle of the black cells in \hat{c} .*

Proof. Initially and for all time $t \geq 0$, $\Phi(\hat{c}^t) \leq 2(n - 2 + m - 2) + A \leq 2N + A$. As long as $\hat{c}^t \neq 0$, $\Phi(\hat{c}^t) > 0$ and according to Lemma 5, $\mathbb{E}[\Delta\Phi_{\delta}(\hat{c}^t)] \leq -1/N$. It follows that the random variable $T = \min\{t : \Phi(\hat{c}^t) \leq 0\}$ is almost surely finite and $\mathbb{E}[T] = O(nA)$ (by applying for example Lemma 2 in [6]); and at that time, \hat{c}^T and c^T are the stable configurations all-white and \blacksquare , respectively. \square

Example 1 (Worst case configurations). Consider the initial dual bounded $n \times n$ -configuration \hat{c} consisting of a black $2 \times (n - 2)$ rectangle. The expected time needed to erase one complete line of the rectangle is at least $\Omega(nN) = \Omega(AN)$.

4 Concluding remarks

This paper proposes an extension to 2D cellular automata of the techniques based on random walks developed in [6,7] to study 1D asynchronous elementary cellular automata. Our techniques apply as well with some important new ingredients, to the Moore neighborhood where the cell fired updates to the minority state within its height closest neighbors [17]. We believe that these techniques may extend to the wide class of threshold automata, which are of particular interest, in neural networks for instance. We are currently investigating refinements of the tools developed here, based on the study of the boundaries between arbitrary checkerboard regions in order to try to prove that every *arbitrary* $n \times m$ -configuration converges to a stable configuration in a polynomial number of steps when n and m are both even (we conjecture a convergence in time $O(N^3)$ for non-bounded toric configurations of even dimensions). This result would conclude the study of this automaton under fully asynchronous dynamics. The experiments lead in Section 2 exhibit an impressive richness of behavior for this yet apparently simple transition rule. An extension of our results to arbitrary α -asynchronous regime is yet a challenging goal, especially if one considers that most of the results concerning spin systems or lattice gas (at the equilibrium) apply only to the limit when the temperature tends to 0, *i.e.*, when only one transition occurs at a time.

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