Full Order Observer with Unmatched Constraint: Unknown Parameters Identification

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Abstract— **This paper concerns both state estimation and parameters identification for linear system with unmatched unknown parts. It deals with a full order delayed unknown inputs observer (DUIO), in which, the time delay concept is investigated to define a new augmented dynamic system including delayed state and output vectors. This estimation approach allows to recover the matching condition which can appear in the observer design problem. The resulting observer has been improved, from the restrictive decoupling condition point of view to guarantee the estimation of state and parameters with asymptotic convergence. Finally, a simulation example based on parametric identification is provided to highlight the feasibility of the suggested method.**

I. INTRODUCTION

The growing need of safety and reliability in many active research areas: control design, process identification and fault detection, has given rise to serious open questions related to estimation problem which can bring undesirable effect on the fulfillment and performance of controlled systems. Throughout the last half century, estimation theory has continuously evolved since the state Luenberger's observer, for linear systems [1], to recent state and unknown input observers (UIO) for nonlinear systems [2]. It is commonly known that a physical system may include unknown parts and subjected to various intrinsic parameters and external perturbations. Designing a virtual sensor for these systems has got significant consideration.

In [3], a reduced-order, minimal-order and full-order observers for a class of nonlinear singular systems have been presented, where the necessary and sufficient conditions for the existence of the observer are established. A finite-time state observer for linear time invariant systems, is presented in [4] under the condition of strong observability and the knowledge of an upper bound of the unknown inputs. A new formulation of such observer, by introducing high order sliding mode, is exposed in [5] where, the sufficient and necessary conditions of strong observability and detectability are formulated in the terms of the system relative degree with respect to unknown inputs. In addition, an extensive use of polytopic and Takagi-Sugeno representation is undertaken, which gives rise to an ease transposition of the over-mentioned observation techniques for nonlinear systems [6]. In [7] the author presented an observer-based fuzzy adaptive event-triggered control studied for a class of pure-feedback nonlinear systems. Fuzzy logic systems are adopted to approximate unknown smooth functions and a fuzzy state observer is designed to estimate unmeasured states. We appreciate the contribution provided in this paper. This paper is considered as a reference on observer design for general nonlinear systems.

Withal these approaches, several open topics need more thorough investigations as for structural constraints [8]. In almost real systems subjected to unknown inputs, parameters and/or disturbances, the matching condition does not hold every time [9]. Overall, research on the problems of mismatched rank condition can primarily be decomposed into two categories: the first one deal with compensation strategies for the effects of unknown model uncertainties and external disturbances, based on motion control systems [10]. The disturbance observer-based control is a practical method in a closed-loop system to compensate matched disturbances. However, it is sensitive to mismatched ones. Alternative methods are proposed based on control framework for perturbation attenuation with unsatisfied matching condition [11]. However, the unmatched unknown part is supposed to be constant, which is restrictive and not always satisfied in real systems. Another set of research is devoted to estimation problem with unmatched unknown inputs [12], considering auxiliary outputs, derived from high-order differentiators of outputs. The effectiveness of using high order sliding mode to overcome the restriction of the matching condition has been proven for various applications mainly for mechanical systems [13]. Nevertheless, to our best knowledge, the convergence of this approach jointly with a parametric identification algorithm needs more deep exploration [14]. In this context, the observer matching constraint is a challenging issue. The main idea of this paper is to propose a new design for the unknown input observer (UIO) to handle the case where all unknown inputs are not decoupled. Our aim is not only to estimate unknown inputs, as in the case of classical UIO, but also to simultaneously identify some system's parameters of interest. The main contributions are:

- An observer-based parametric identification is proposed.
- The proposed observer allows the simultaneous identification of a set of constant parameters without using a third-party classical identification algorithm.
- A more relaxed method to satisfy rank condition since it uses time delayed states rather than a third-party differentiators which add more restrictive condition to ensure convergence.
- The proposed observer gives a general framework to define a step by step algorithm for DUIO observer.

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The remainder is organized as follows. In the next section, a class of systems and the problem statement are introduced. A technical concept based delayed states and outputs vectors is given in section III, to define an augmented model. Section IV, is devoted to the main contribution of the paper, namely the delayed unknown inputs observer design and the convergence analysis. In section V, the effectiveness of the proposed observer is highlighted via simulation results involving a robustness test. Finally, some concluding remarks are given in section VI.

II. Problem Statement

Consider the following continuous Linear Time-Invariant (LTI) system with a part affine to unknown parameters:

$$\begin{cases} \dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) + \bar{D}_{\bar{y}}\theta \\ \bar{y} &= \bar{C}\bar{x}(t) \end{cases} \tag{1}$$

where $\bar{x}(t) \in \mathbb{R}^n$ is the state vector of the system, $\bar{u}(t) \in \mathbb{R}^{n_{\bar{u}}}$ be the input vector and $\bar{y}(t) \in \mathbb{R}^{n_{\bar{y}}}$ the measured output vector, and $\theta \in \mathbb{R}^{n_\theta}$ represents the constant unknown parameters $\dot{\theta} = 0$. The matrix $\bar{D}_{\bar{y}} = \bar{D}(\bar{y}) \in \mathbb{R}^{n \times n_\theta}$ represents a time-varying dynamics of the plant. The matrices $\bar{A} \in \mathbb{R}^{n \times n}$, $\bar{C} \in \mathbb{R}^{n_{\bar{y}} \times n}$ and $\bar{B} \in \mathbb{R}^{n \times n_{\bar{u}}}$ are considered constants. Without loss of generality, one consider that $\text{rank}(\bar{C}) = n_{\bar{y}}$ and $\text{rank}(\bar{D}_{\bar{y}}) = n_\theta$, $\forall \bar{y} \in \Delta$, where Δ defines hyper-rectangles:

$$\boldsymbol{\Delta} = \left\{ \bar{y}, \dot{\bar{y}} \in \mathbb{R}^{n_{\bar{y}}} \middle| \ \bar{y}_{i_{min}} \leq \bar{y}_i \leq \bar{y}_{i_{max}}, \ \dot{\bar{y}}_{i_{min}} \leq \dot{\bar{y}}_i \leq \dot{\bar{y}}_{i_{max}} \ \right\} \tag{2}$$

The observer design approach is based on the existence of the left inverse matrix, called decoupling matrix, to make possible the reconstruction for the unknown input [4]. In other words, this condition imposes that the number of unknown parameters must be less than the number of system's outputs. The rank condition is given as following:

$$rank(\bar{C}\bar{D}_{\bar{y}}) = rank(\bar{D}_{\bar{y}}) \tag{3}$$

Also, we have $(\bar{C}\bar{D}_{\bar{y}}) \in \mathbb{R}^{n_{\bar{y}} \times n_\theta}$ and $(\bar{D}_{\bar{y}}) \in \mathbb{R}^{n \times n_\theta}$. By computing the rank of each part we have:

$$rank(\bar{D}_{\bar{y}}) = \min(n, n_\theta) = n_\theta \tag{4}$$
$$rank(\bar{C}\bar{D}_{\bar{y}}) = \min(n_{\bar{y}}, n_\theta) = n_{\bar{y}} \tag{5}$$

If $n_\theta > n_{\bar{y}}$ then the rank condition is not fulfilled which is the case of the original system: $rank(\bar{C}\bar{D}_{\bar{y}}) \neq rank(\bar{D}_{\bar{y}})$.
To overcome this restriction, the original system must be augmented by introducing auxiliary outputs. Almost methods use differentiators to get a successive time-derivatives of the original system's outputs. In our method, a more relaxed approach is proposed by taking the time- delayed outputs of the original system's to fulfill the previous rank condition.

Throughout this paper, let us adopt the matrices notation $*_{\bar{y}} = *(\bar{y}(t))$, $*_{\bar{y},\dot{\bar{y}}} = *(\bar{y}(t), \dot{\bar{y}}(t))$, and $*_{\tau_i} = *(t - \tau_i)$, $*_{\bar{y}_{\tau_i}} = *(\bar{y}(t - \tau_i))$, where τ_i is a constant delay. $0_{p \times q}$ is null matrix of p lines and q column and $I_{p \times q}$ stands for an identity matrix of p lines and q column and $\mathbb{R}$ represents the set of real numbers.

III. Augmented State Space

This section concerns the modelling transformation to break out with the rank condition related to mismatched unknown part. An augmented system is considered by including delayed states and outputs dynamics. Considering delays τ_i $(1 \leq i \leq m)$, the output vector $\bar{y}$ and $\bar{y}_\tau$ exist, an augmented model is constructed with a new state, input and output vectors $x(t)$, $u(t)$ and $y(t)$:

$$\begin{aligned} x &= [\ \bar{x}^T \quad \bar{x}^T_{\tau_1} \quad \bar{x}^T_{\tau_i} \quad \dots \quad \bar{x}^T_{\tau_m}\ ]^T \\ y &= [\ \bar{y}^T \quad \bar{y}^T_{\tau_1} \quad \bar{y}^T_{\tau_i} \quad \dots \quad \bar{y}^T_{\tau_m}\ ]^T \\ u &= [\ \bar{u}^T \quad \bar{u}^T_{\tau_1} \quad \bar{u}^T_{\tau_i} \quad \dots \quad \bar{u}^T_{\tau_m}\ ]^T \end{aligned} \tag{6}$$

where τ_i is the i^{th} time delay, and i represents the number of sub-systems to add. The parameter m is an integer computed using the following condition:

$$\frac{n_\theta}{n_{\bar{y}}} - 1 \leq m < \frac{n_\theta}{n_{\bar{y}}} \tag{7}$$

where m represents the number of set of auxiliary outputs to add in order to recover the unmatched condition.
The augmented state-space system is given by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + D_y\theta \\ y = Cx(t) \end{cases} \tag{8}$$

where,

$$A = \begin{bmatrix} \bar{A} & 0_n & \cdots & 0_n \\ 0_n & \bar{A} & \cdots & 0_n \\ \vdots & \vdots & \ddots & \vdots \\ 0_n & 0_n & \cdots & \bar{A} \end{bmatrix} B = \begin{bmatrix} \bar{B} & 0_{n \times n_{\bar{u}}} & \cdots & 0_{n \times n_{\bar{u}}} \\ 0_{n \times n_{\bar{u}}} & \bar{B} & \dots & 0_{n \times n_{\bar{u}}} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n \times n_{\bar{u}}} & 0_{n \times n_{\bar{u}}} & \cdots & \bar{B} \end{bmatrix}$$

$$D_y = \begin{bmatrix} \bar{D}_{\bar{y}} \\ \bar{D}_{\bar{y}_{\tau_1}} \\ \vdots \\ \bar{D}_{\bar{y}_{\tau_m}} \end{bmatrix} C = \begin{bmatrix} \bar{C} & 0_{n_{\bar{y}} \times n} & \cdots & 0_{n_{\bar{y}} \times n} \\ 0_{n_{\bar{y}} \times n} & \bar{C} & \cdots & 0_{n_{\bar{y}} \times n} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n_{\bar{y}} \times n} & 0_{n_{\bar{y}} \times n} & \cdots & \bar{C} \end{bmatrix}$$

where, $A \in \mathbb{R}^{((m+1) \times n) \times ((m+1) \times n)}$, $B \in \mathbb{R}^{((m+1) \times n) \times ((m+1) \times n_{\bar{u}})}$, $D_y \in \mathbb{R}^{((m+1) \times n) \times n_\theta}$ and $C \in \mathbb{R}^{((m+1) \times n_{\bar{y}}) \times ((m+1) \times n)}$. System (8) is a straightforward generalization to satisfy the matching condition.

A. Geometric conditions of rank matrices

Considering the original system's defined in (1), such that:

$$\begin{cases} n_{\bar{y}} < n \\ \text{rank}(\bar{C}) = n_{\bar{y}} \quad \text{and} \quad \text{rank}(\bar{D}_{\bar{y}}) = n_\theta \\ n_{\bar{y}} < n_\theta \quad \rightarrow \quad rank(\bar{C}) < rank(\bar{D}_{\bar{y}}) \end{cases} \tag{9}$$

Therefore,

$$\text{rank}(\bar{C}\bar{D}_{\bar{y}}) \leq \min(\text{rank}(\bar{C}), \text{rank}(\bar{D}_{\bar{y}})) = n_{\bar{y}} < n_\theta \tag{10}$$

In this case, the observer rank condition is not satisfied, $\text{rank}(\bar{C}\bar{D}_{\bar{y}}) \neq rank(\bar{D}_{\bar{y}})$. The original system is augmented with both the model dynamics and m additional delayed state vectors, defined so that the observer matching condition is met. Therefore, the system's dimension, with respect to equation (7), is as follow:

$$\begin{cases} \text{rank}(C) = (m+1)n_{\bar{y}} \\ n_\theta \leq (m+1)n_{\bar{y}} < (m+1)n \\ \text{rank}(D_y) = \min((m+1)n, n_\theta) = n_\theta \end{cases} \tag{11}$$

Also:

$$\begin{aligned}\text{rank}(CD_y) &\leq \min(\text{rank}(C),\text{rank}(D_y)) = \min((m+1)n_{\bar{y}}, n_\theta) \\ &\Rightarrow \text{rank}(CD_y) \leq n_\theta \end{aligned} \quad (12)$$

One can notice that the matrix $\text{rank}(D_y) = \text{rank}(\bar{D}_{\bar{y}}) = n_\theta$ keep the same number of column because of: $dim(\theta) = n_\theta$ is constant on the original and augmented models. Knowing that the decoupling matrix CD_y has non collinear rows, such that $(n_\theta \leq (m+1)n_{\bar{y}})$. The idea is to enhance the rank of the decoupling matrix in the augmented model such that $n_\theta \leq (m+1)n_{\bar{y}}$, to fulfilled the matching condition imposed by the observer, it implies:

$$rank(CD_y) = \min((m+1)n_{\bar{y}}, n_\theta) = n_\theta = rank(D_y) \quad (13)$$

B. *Observability Analysis*

The state and parameter reconstruction problem is closely linked to the problem of observability, as shown in [15]. Therefore, we recall some important definitions about strong observability and strong detectability of systems with unknown parameters. Consider the following system with $x(t)$ is the state vector, $u(t)$ is the known inputs vector, θ the unknown parameters vector and $y(t)$ is the measurements vector:

$$\begin{cases} \dot{x}(t) &= f(x(t),u(t),\theta) \\ y(t) &= h(x(t),u(t),\theta) \end{cases} \quad (14)$$

Definition 1: [16] For every initial condition $x(0)$, any known input $u(t)$ and any couple of unknown parameters $(\theta,\bar{\theta})$, the system (14) with two different trajectories $x(t)$ and $\bar{x}(t)$ is called:

- state and unknown parameters strongly observable: if $y(t,x(t),u(t),\theta)$= $y(t,\bar{x}(t),u(t),\bar{\theta})$ implies that: $x(t) = \bar{x}(t)$ and $\theta = \bar{\theta}$.
- state and unknown parameters strongly detectable: if $y(t,x(t),u(t),\theta)$= $y(t,\bar{x}(t),u(t),\bar{\theta})$ implies that: $x(t) \to x(t)$ and $\theta \to \theta$ as $t \to \infty$.

Definition 1 concerns the state and parameters observability or detectability. The unknown parameters observability (detectability) relates to the possibility of reconstruct the unknown part uniquely infinite-time (asymptotically) having as information the known inputs and outputs.

Assumption 1: Augmented system (A,D_y,C) is assumed to be uniformly strongly observable (Definition 1).

IV. Observer Design

The problem of Full Order Delayed Unknown Input Observer (DUIO) design can be stated as follows:

$$\begin{cases} \dot{z}(t) &= N_{y,\dot{y}}z(t) + L_{y,\dot{y}}y(t) + G_y u(t) \\ \hat{x}(t) &= z(t) - H_y y(t) \end{cases} \quad (15)$$

where, $\hat{x}$ and $\hat{y}$ are the estimated state and the output vector, respectively. The matrices $N_{y,\dot{y}} \in \mathbb{R}^{((m+1)\times n)\times((m+1)\times n)}$, $L_{y,\dot{y}} \in \mathbb{R}^{((m+1)\times n)\times((m+1)\times n_{\bar{y}})}$, $H_y \in \mathbb{R}^{((m+1)\times n)\times((m+1)\times n_{\bar{y}})}$ and $G_y \in \mathbb{R}^{((m+1)\times n)\times((m+1)\times n_{\bar{u}})}$ are parameter varying, well-posed to satisfy a stable asymptotic convergence of the estimation error dynamics. The proposed estimation approach is taken under the fulfilled matching condition in the augmented model.

Definition 2: The DUIO observer is called asymptotically stable if the observer error $\tilde{x}(t) = x(t) - \bar{x}(t)$ converges to zero when $t \to \infty$: $\lim_{t\to\infty} \tilde{x}(t) = 0$. Indeed, the continuous-time error dynamics satisfies $\dot{\tilde{x}}(t) = N_{y,\dot{y}}\tilde{x}(t)$ when the matrix $N_{y,\dot{y}}$ has all the eigenvalues inside the unit circle (i.e $N_{y,\dot{y}}$ Hurwitz). Assume that there exists a matrix P_y defining as $P_y = I_{(m+1)\times n} + H_y C$. Now, from the augmented model (8) and DUIO (15), one can easily prove that the state estimate error $\tilde{x} = x - \hat{x}$, is given by:

$$\tilde{x} = P_y x - z \quad (16)$$

The derivative of the state estimation error yields:

$$\begin{aligned} \dot{\tilde{x}} &= \dot{P}_{y,\dot{y}}x + P_y\dot{x} - \dot{z} \\ &= N_{y,\dot{y}}\tilde{x} + (\dot{P}_{y,\dot{y}} + P_y A - N_{y,\dot{y}}P_y - L_{y,\dot{y}}C)x \\ &+ P_y D_y \theta + (P_y B - G_y)u \end{aligned} \quad (17)$$

Obviously, if the ordinary differential equation of estimation errors is defined as $\dot{\tilde{x}}(t) = N_{y,\dot{y}}\tilde{x}(t)$ with, $N_{y,\dot{y}}$ is a Hurwitz matrix. Therefore the designed estimator can proceed to ensure asymptotic estimates for system states. Let us now establish the following specified conditions for the asymptotic convergence:

(i) $\dot{P}_{y,\dot{y}} + P_y A - N_{y,\dot{y}}P_y - L_{y,\dot{y}}C = 0$.
(ii) $P_y D_y = 0$
(iii) $P_y B - G_y = 0$.

Substituting these conditions into (17). The estimation error dynamics will be reduced to:

$$\dot{\tilde{x}}(t) = N_{y,\dot{y}}\tilde{x}(t) \quad (18)$$

The observer gains that satisfy the convergence conditions, can be obtained by the following steps:

- Knowing that $P_y = I_{(m+1)\times n} + H_y C$, condition (ii) leads to:

$$P_y D_y = 0 \Leftrightarrow H_y = -D_y\left[(CD_y)^T(CD_y)\right]^{-1}(CD_y)^T \quad (19)$$

- After computing H_y, one obtain:

$$P_y = I_{(m+1)\times n} + H_y C \quad (20)$$

- From condition (iii), one has:

$$G_y = P_y B \quad (21)$$

Note that the matrices H_y, P_y and G_y are determined from the model matrices D_y and C.

- From condition (i), $N_{y,\dot{y}}$ can be expressed as:

$$N_{y,\dot{y}} = \Gamma_{y,\dot{y}} - K_{y,\dot{y}}C \quad (22)$$

where,

$$\Gamma_{y,\dot{y}} = \dot{P}_{y,\dot{y}} + P_y A, \qquad K_{y,\dot{y}} = N_{y,\dot{y}}H_y + L_{y,\dot{y}} \quad (23)$$

Therefore, the dynamical estimation error becomes:

$$\dot{\tilde{x}}(t) = (\Gamma_{y,\dot{y}} - K_{y,\dot{y}}C)\tilde{x}(t) \quad (24)$$

A. Polytopic form

The Takagi-Sugeno (TS) representation is undertaken [6], giving rise to an ease transposition of the over-mentioned time varying matrices. It will be assumed that, the premise variables $y, \dot{y} \in \mathbf{\Delta}$ are real-time accessible. The $2n_{\bar{y}}$ nonlinearities related to $y, \dot{y} \in \mathbf{\Delta}$ are captured via membership functions $\eta_i(\cdot)$, which have the convex-sum property in the compact set of the state space:

$$\sum_{i=1}^{r} \eta_i(y, \dot{y}) = 1, \quad 0 \leq \eta_i(y, \dot{y}) \leq 1 \tag{25}$$

While $r = 2^{2n_{\bar{y}}}$ is the number of the sub-models.

The TS representation is considered only for defining the Linear Matrix Inequality (LMI) at the end, in order to compute the observer gains. Then, one obtains the polytopic exact forms:

$$\Gamma_{y,\dot{y}} = \sum_{i=1}^{r} \eta_i(y,\dot{y})\Gamma_i \quad , \qquad K_{y,\dot{y}} = \sum_{i=1}^{r} \eta_i(y,\dot{y})K_i \tag{26}$$

where, Γ_i and K_i are constant matrices. From this representation, the gain matrices $L_{y,\dot{y}}$ and $N_{y,\dot{y}}$ can be defined as:

$$\begin{cases} N_{y,\dot{y}} = \sum_{i=1}^{r} \eta_i(y,\dot{y})N_i, & N_i = \Gamma_i - K_iC \\ L_{y,\dot{y}} = \sum_{i=1}^{r} \eta_i(y,\dot{y})L_i, & L_i = K_i - N_iH_i \end{cases} \tag{27}$$

Consequently, the polytopic transformation of the known matrices leads to the following state estimation error dynamics:

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{r} \eta_i(y,\dot{y})\,(\Gamma_i - K_iC)\,\tilde{x}(t), \; i \in = 1,2,...,2^{2n_{\bar{y}}} \tag{28}$$

B. Convergence analysis

The stability analysis of the estimation problem has highlighted significant connections with Lyapunov theory. The following theorem is given to define the LMI conditions of the existence of the observer.

Theorem 1: The full order delayed unknown input observer (15) for the augmented model (8), guaranties the state estimation convergence, if there exists a symmetric positive definite matrix $Q \in \mathbb{R}^{(m+1)n \times (m+1)n}$ defining a Lyapunov function $V(\tilde{x}) > 0$, such that $\dot{V}(\tilde{x}) < 0$, $\forall \tilde{x}(t) \neq 0$. Hence, the following linear matrix inequality holds:

$$\begin{pmatrix} -Q & 0 \\ 0 & \Gamma_i^T Q + Q\Gamma_i - C^T {R_i}^T - R_iC \end{pmatrix} < 0 \tag{29}$$

Proof 1: The observer gains are selected so that N_i is a Hurwitz matrix, based on the stability analysis of the Lyapunov theory. Now, consider that there exists a positive definite matrix function $Q = Q^T > 0 \in \mathbb{R}^{((m+1)\times n)\times((m+1)\times n)}$ such that a quadratic Lyapunov function is defined to analyse the asymptotic convergence of the dynamical error, as follow:

$$V(\tilde{x}) = \tilde{x}^T Q \tilde{x} \tag{30}$$

Taking time derivative of $V(\tilde{x})$ along the error dynamics yields:

$$\dot{V} = \tilde{x}^T \left(\sum_{i=1}^{r} \eta_i(y,\dot{y}) \left(\Gamma_i^T Q + Q\Gamma_i - C^T {K_i}^T Q - QK_iC \right) \right) \tilde{x} \tag{31}$$

Note that $\dot{V} < 0$, implies that the estimation error $\tilde{x}(t)$ tends to zero asymptotically for any initial value $\tilde{x}(0)$. It follows that the bilinear matrix inequality (BMI) holds:

$$\Gamma_i^T Q + Q\Gamma_i - C^T {K_i}^T Q - QK_iC < 0, \; Q = Q^T > 0 \tag{32}$$

One note that the inequality (32) is bilinear with respect to the unknown matrices Q and K_i. The solution of this matrix inequality is quite different. To solve this problem, one can consider the following change of variable: $R_i = QK_i$. Therefore, the following linear matrix inequality holds:

$$\Gamma_i^T Q + Q\Gamma_i - C^T {R_i}^T - R_iC \; < 0 \tag{33}$$

Thus, from the Lypunov stability theory, if the LMI condition (29) is satisfied, the system (15) is exponentially asymptotically stable. This completes the proof of Theorem 1.

C. Parameters estimation

In this subsection, the unknown parameters are identified by considering the derivatives of the output vector of equation $\hat{y} = C\hat{x}$. By algebraic inversion of the derivatives equation, one can reconstruct unknown parameters vector $\hat{\theta}$ from the estimation of the state vector and output derivatives, as follow:

$$\hat{\theta} = \left[(CD_y)^T (CD_y) \right]^{-1} (CD_y)^T \left(\dot{\hat{y}} - CA\hat{x} - CBu \right) \tag{34}$$

However, the feasibility of this inversion is conditioned by a convenient selection of the time delay to fulfil rank condition. Then, parameters identification is done in two steps. In the first step, the state vector are estimated from the equation of DUIO (15) while, the parameter vector is identified in the next step, by substituting output derivatives in the inversion model (34). Where, the parameter estimation error is defined as:

$$e_\theta = \theta - \hat{\theta} = -\left[(CD_y)^T (CD_y) \right]^{-1} (CD_y)^T CA\tilde{x} \tag{35}$$

From (35), it is easy to show that the convergence of $\hat{\theta}$ towards θ is ensured from the asymptotic stability of state estimation errors $\tilde{x}(t)$ under suitable persistence excitation.

Let us reformulate the excitation conditions ensuring parameter convergence of the estimates $\hat{\theta}$ towards θ.

Definition 3: The Persistent Excitation Condition is obtained if there exist $c_{1_{ij}}$, $c_{2_{ij}}$ and $c_{3_{ij}}$ for $i, j = 1, ..., q$, such that for all t the following inequality holds [17]:

$$c_{1_{ij}} I \leq \int_{t_0}^{t_0 + c_{3_{ij}}} \bar{D}_{\bar{y}_{\tau_i}} \bar{D}_{\bar{y}_{\tau_j}}^T dt \quad \leq c_{2_{ij}} I \quad \forall i, j = 1, ..., q$$

The design procedure of the DUIO observer can be summarized in the following design Algorithm 1.

V. Motivating Example

In this section, an example is proposed to illustrate the estimation performance of DUIO observer. Consider the following matrices for the LTI system (1):

$$\bar{A} = \begin{bmatrix} -30 & 12 & -20 \\ -50 & -13 & 0 \\ 10 & 2 & -14 \end{bmatrix}, \; \bar{B} = \begin{bmatrix} 1.5 \\ 0 \\ 2 \end{bmatrix}, \tag{36}$$

$$\bar{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}, \; \bar{D}_{\bar{y}} = \begin{bmatrix} y_2 & 0 & 0 \\ 0 & 0 & y_1 \\ 0 & y_1 & y_2 \end{bmatrix}, \; \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Algorithm 1 Observer design procedure

1: **procedure** DUIO
2: Check if system (A, D_y, C) is observable or detectable. If so, go to Step 3; Otherwise, stop.
3: Check the decoupling conditions :
4: **if** $rank(\bar{C}\bar{D}_y) = rank(\bar{D}_y)$ $\forall y \in \boldsymbol{\Delta}$ hold **then**:
5: $m = 0$, go to step 11.
6: **end if**
7: **if** $rank(\bar{C}\bar{D}_y) \neq rank(\bar{D}_y)$ **then**:
8: Find the minimum integer $(m) \leftarrow$ according to (7).
9: Find the augmented matrices A, B, C, D in (8), go to the next step;
10: **end if**
11: Compute matrices H_y in (19), P_y in (20) and deduce the matrix G_y in (21).
12: Compute the matrix $\dot{P}_{y,\dot{y}}$ and deduce $\Gamma = P_y A_y + \dot{P}_{y,\dot{y}}$ in polytopic form (23 and 26).
13: Solve the LMIs in (29) for the variables Q, R_i such that N_i is Hurwitz.
14: Compute $K_i = Q^{-1}R_i$ which gives the matrix $K_{y,\dot{y}}$.
15: Deduce $N_{y,\dot{y}} = \Gamma - K_{y,\dot{y}}C$ and $L_{y,\dot{y}} = K_{y,\dot{y}} - N_{y,\dot{y}}H_y$ from equation (32).
16: Construct observer (15) and get the estimations of state $\hat{x}(t)$.
17: Estimate the parameters $\hat{\theta}$ by a simple dynamic system inversion in (34).
18: **end procedure**

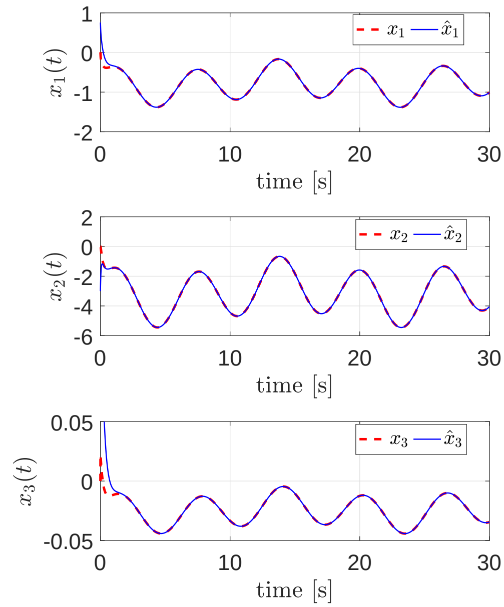

Fig. 2. State Estimation: Actual state (red) their estimates (blue)

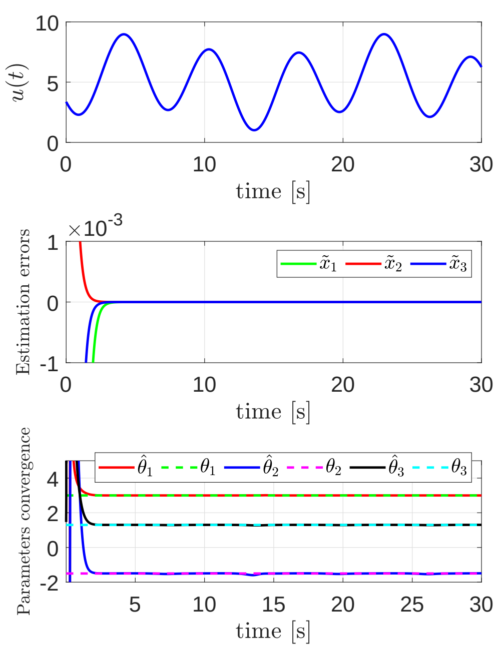

Fig. 1. Input-Estimation Errors- Parameters convergence. The convergence is compared with $RMS = (0.0731, 0.2165, 0.0150)$, $MSE = (0.0053, 0.047, 2.236.10^{-4})$.

For this given example, it is easy to verify that observer existence rank conditions is not fulfilled for the original system (36): $rank(\bar{C}\bar{D}_{\bar{y}}) = 2 \neq rank(\bar{D}_{\bar{y}}) = 3$, $\forall(y_1, y_2) \neq (0,0)$. It means the output vector is less than the unknown parameters: $(n_{\bar{y}}, n_{\theta})$=$(2,3)$.

The original system is augmented with $m = 1$ delay in equation (8). Taking time delay τ_m leads to $dim(y = [\bar{y}, \bar{y}_{\tau_m}]) = 4 > dim(\theta) = 3$. One get: $C_y = \left[\bar{C}, 0; 0, \bar{C}\right]$, $D_y = \left[\bar{D}_{\bar{y}}; \bar{D}_{\bar{y}_{\tau_m}}\right]$. Therefore, the rank condition is fulfilled: rank$(CD_y) = rank(D_y) = 3$. The design of the DUIO observer has been achieved by considering the optimization problem under LMI constraints. Thus, the solution of theorem (29) is computed by using the procedure Yalmip of the Matlab LMI Toolbox [18]. Due to space limitation the observer gain are not given. Since asymptotic convergence, unknown parameters are estimated from model inversion 34 assuming that all the states are available (either measurable or estimated). The initial condition are: $\bar{x}_0 = \bar{x}_{\tau_{m_0}} = [0,0,0]^T$, $\hat{\bar{x}}_0 = \hat{\bar{x}}_{\tau_{m_0}} = [0.75, -3, 0.15]^T$, $\tau_m = 0.5(sec)$.

A. *The noise-free case*

The first case is studied to illustrate the estimation performance and to corroborate the theoretical results obtained through the convergence analysis carried in the above sections. In this case, the input vector $u(t)$, state estimation error $\tilde{x}(t)$ and the convergence of the estimated parameters, compared to nominal values are presented in Fig.1. the variations noted in the proximity of the time origin are due to the initial conditions of the observer which have been arbitrarily chosen. Fig. 2 compares the states reconstructed by the DUIO observer with the corresponding states of system. The observer states starts from the initial condition and converge to the system state, the quality of the reconstruction is highlighted under $RMS = \sqrt{\frac{\sum_{i=1}^{n}(x(i)-\hat{x}(i))^2}{n}}$ and $MSE = mean(\sum_{i=1}^{n}(x(i) - \hat{x}(i))^2)$ to illustrate the convergence in the estimated states. The simulation results show that, the DUIO observer rapidly and accurately estimates the state of the model even if initial conditions are not the same. Moreover, parameters estimations lead to a peaking phenomenon in which initial estimator error can be prohibitively large, then, these parameters converge to true value after peaking has subsided. As a consequence, with small initial conditions, observer converge quickly to brings the estimated state's error to zero in small time.

B. *The noisy outputs case*

During the design process of the observer, no perturbation was considered. In order to simulate practical situations, one considers the system affected by noise to test the robustness of the estimation approach. The outputs issued from the simulation of system are corrupted by additive noise bounded by a ratio around of 5%. Figure (3) shows the error in the estimated state, and the parameters convergence, both figures exhibit noisy case. Notice that the unknown parameters in figure 3 are estimated with a good way even if we considered perturbed measures. One can remove the noise effect in parameters with a simple low-pass filter. Simulation result, shows that the DUIO observer is robust enough to handle the noisy case.

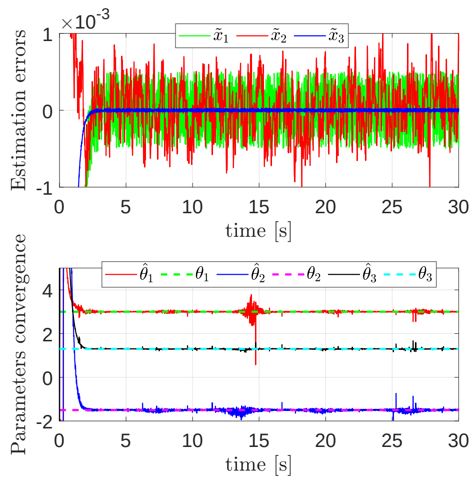

Fig. 3. The noise-corrupted case

Summarizing this section, the estimation approach gives a solution even if the decoupling matrix is not of full rank in the original system. It can be appreciated that the observer performs as expected and the state errors reach zero asymptotically. This result proves the reliability of the approach to reconstruct unmeasured states and identify the unknown parameters.

VI. Conclusions

The paper shows some significant features, to discuss how a failed decoupling condition, can be recovered using augmented model and time delay concept, with a specific characterization of the system matrices. Based on these last, a step by step algorithm is developed to design the DUIO observer. Different from existing results, the proposed observer gives a general framework for observer-based parameters identification with arbitrary relative degree with respect to unknown parameters. The simulation results are quite promising to prove that the estimation approach provides an interesting solution for state reconstruction and parameters identification.

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