The Busy Beaver Game for Reaction Systems

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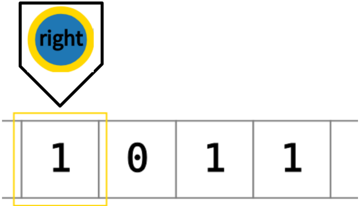
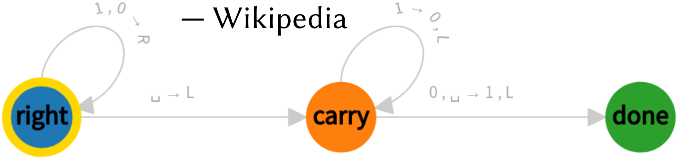
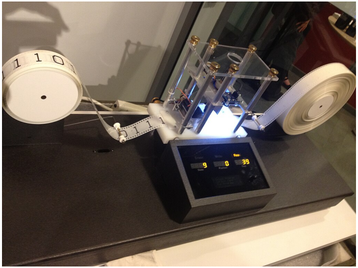
The Turing machine

“an abstract machine that manipulates symbols on a strip of tape according to a table of rules”

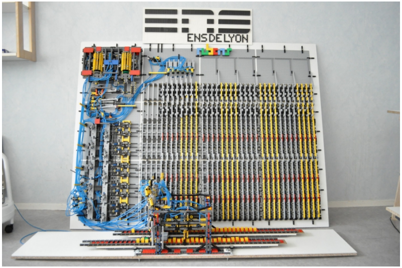
— Wikipedia







https://aturingmachine.com/

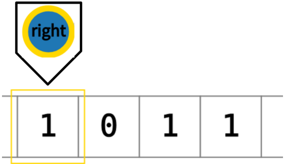


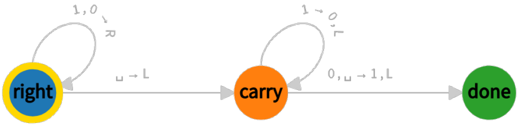
http://rubens.ens-lyon.fr/

The busy beaver game

$BB(n, m)$ = max number of steps taken by a halting n-state, m-symbol Turing machine starting from all 0.

— bbchallenge.org





$n = 3$ states

$m = 2$ symbols

Binary increment ⇒ not a champion : $BB(2, 3) = 38$.

Reaction systems

alphabet of symbols
finite background set of species

inhibitors ↘

$$a : (R_a, I_a, P_a)$$

↗ reactants ↖ products

$$R_a, I_a, P_a \subseteq S$$

a is enabled in $W \subseteq S$ iff $R_a \subseteq W$ and $I_a \cap W = \emptyset$.

Application: $res_a(W) = P_a$.

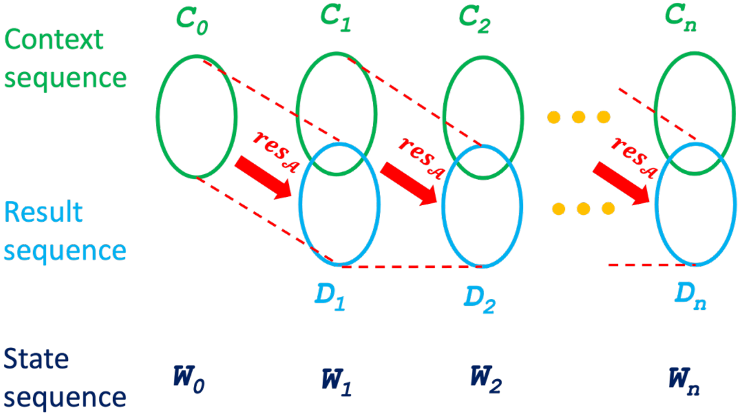
- a species is either absent or present in ∞ amounts
- species disappear by default

Reaction system: $\mathscr{A} = (S, A = \{a_1, a_2, \dots\})$

↑ finite set of reactions

Interactive processes

Driven by a context sequence: $C_0, C_1, \ldots, C_n \subseteq S$.



needs context to run

— www.reactionsystems.org

$$W_{i+1} = res_{\mathscr{A}}(C_i \cup D_i)$$

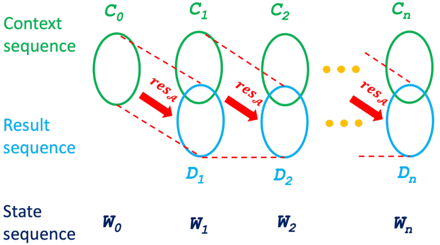
Reaction systems only have $2^{|S|}$ states.

The busy beaver game for reaction systems ??

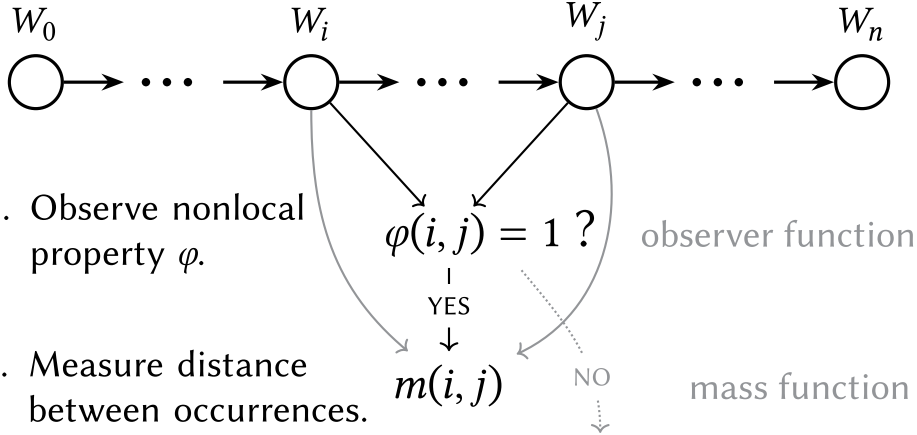
Generalized busy beaver game

The plan

Observe nonlocal properties on an interactive process.



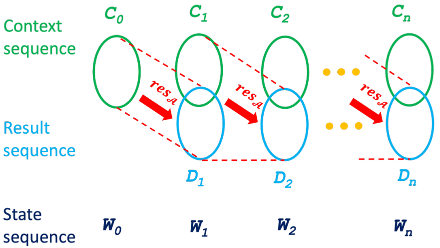
1. Observe nonlocal property φ.
2. Measure distance between occurrences.

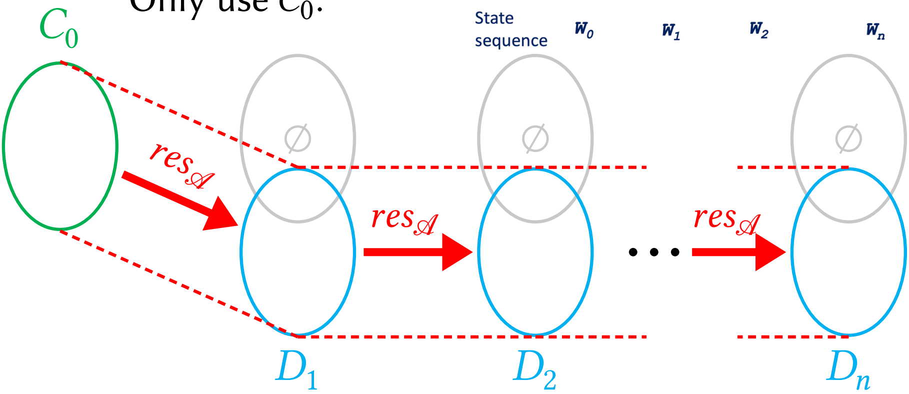


Input $\mapsto$ Seed

Set $\forall i > 0 \,:\, C_i = \varnothing$.

Only use C_0.





$\sigma(C_0)$: all seeds obtained from C_0

- all context sequences $(C_0, \varnothing, \dots, \varnothing)$ of any length

Busy beaver frame

Collects the essential ingredients for defining the generalized busy beaver game.

partially ordered $\preceq$

$$\mathcal{F} = (\varphi, m, M, \mathrm{size}, \mathbf{n})$$

observer function

mass function

domain of m

size function

target size $\mathbf{n} \in \mathbb{N}$

$m : \mathbb{N}^k \to (M, \preceq)$

$\mathrm{size} : \mathbb{A} \to \mathbb{N}$

$\uparrow$

all reaction systems

A note on size

all reaction systems

$$\downarrow$$

$$\text{size} : \mathbb{A} \to \mathbb{N}$$

Let $\mathscr{A} = (S, A) \in \mathbb{A}$. Multiple options for size:

- $\text{size}(\mathscr{A}) = |S|$ number of species
- $\text{size}(\mathscr{A}) = |A|$ number of reactions
- $\text{size}(\mathscr{A}) = \max\{ |R_a|, |I_a|, |P_a| \mid a = (R_a, I_a, P_a) \in A \}$
- ...

Generalized busy beaver game

read this way

$$\mu_{\mathcal{F}} = \max_{\mathcal{A}\in\mathcal{R}\mathcal{S}(\mathbf{n})} \ \max_{C_0\subseteq S_{\mathcal{A}}} \ \min_{\gamma\in\sigma(C_0)} \{m(\lambda) \mid \lambda \in \varphi(\pi_\gamma)\}$$

compute this way

Generalized busy beaver game

read this way →

the interactive process driven by γ

positions at which φ is observed

a measure on λ

$$\mu_{\mathcal{F}} = \max_{\mathscr{A} \in \mathscr{R}\mathscr{S}(\mathbf{n})} \max_{C_0 \subseteq S_{\mathscr{A}}} \min_{\gamma \in \sigma(C_0)} \{m(\lambda) \mid \lambda \in \varphi(\pi_\gamma)\}$$

a reaction system of size $\mathbf{n}$

a context of $\mathscr{A}$

a seed starting with C_0

compute this way ←

$\mu_{\mathcal{F}}$ breakdown

because reaction systems don't halt

1. Fix $\mathscr{A}$ and a C_0, minimize m over $\sigma(C_0)$:

$$\mu_{\mathcal{F}}(\mathscr{A}, C_0) = \min_{\gamma \in \sigma(C_0)} \{m(\lambda) \mid \lambda \in \varphi(\pi_\gamma)\}$$

2. For $\mathscr{A}$, take C_0 maximizing $\mu_{\mathcal{F}}(\mathscr{A}, C_0)$:

$$\mu_{\mathcal{F}}(\mathscr{A}) = \max_{C_0 \subseteq S_{\mathscr{A}}} \{\mu_{\mathcal{F}}(\mathscr{A}, C_0)\}$$

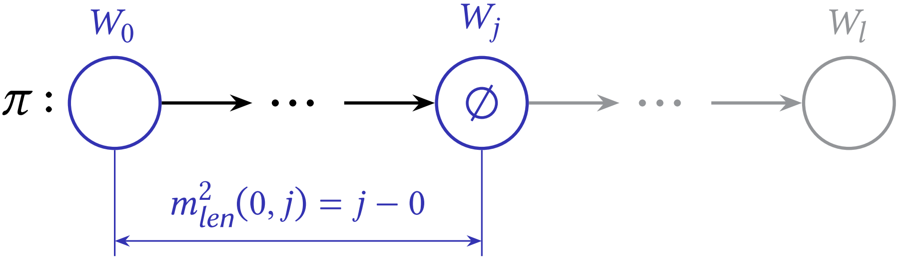
3. Pick $\mathscr{A}$ maximizing $\mu_{\mathcal{F}}(\mathscr{A})$: the busy beaver champion

$$\mu_{\mathcal{F}} = \max_{\mathscr{A} \in \mathscr{RS}(\mathbf{n})} \mu_{\mathcal{F}}(\mathscr{A})$$

Concrete busy beaver games

Longest runs to ∅

because reaction systems don't halt.



$$\mathcal{F}_{lr}(\text{size}, \mathbf{n}) = (\varphi_{lr}, m^2_{len}, \mathbb{Z}, \text{size}, \mathbf{n})$$

$$\varphi_{lr}(\pi, i, j) = \begin{cases} 1, & i = 0 < j \text{ and } W_j = \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

$\mathscr{F}_{lr}$ breakdown

1. Find the shortest time it takes $\mathscr{A}$ to get from C_0 to $\emptyset$:

$$\mu_{\mathscr{F}_{lr}}(\mathscr{A}, C_0) = \min_{\gamma \in \sigma(C_0)} \{m^2_{len}(\lambda) \mid \lambda \in \varphi_{lr}(\pi_\gamma)\}$$

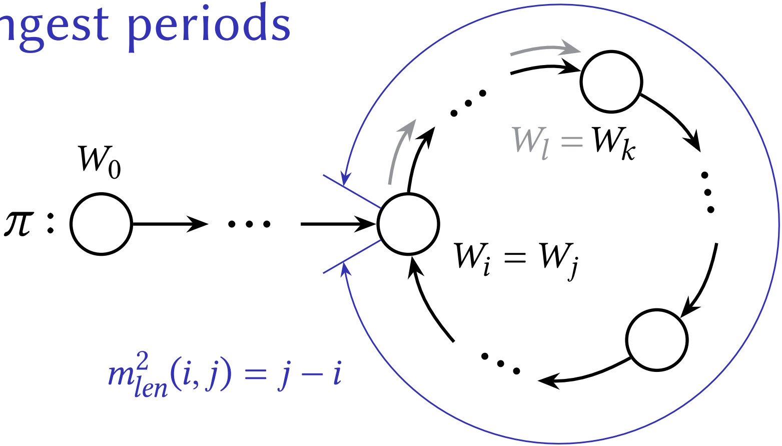
2. Find C_0 yielding the longest shortest time to $\emptyset$:

$$\mu_{\mathscr{F}_{lr}}(\mathscr{A}) = \max_{C_0 \subseteq S_{\mathscr{A}}} \{\mu_{\mathscr{F}_{lr}}(\mathscr{A}, C_0)\}$$

3. Find $\mathscr{A}$ showing the largest longest shortest time to $\emptyset$:

$$\mu_{\mathscr{F}_{lr}} = \max_{\mathscr{A} \in \mathscr{RS}(\mathbf{n})} \mu_{\mathscr{F}_{lr}}(\mathscr{A})$$

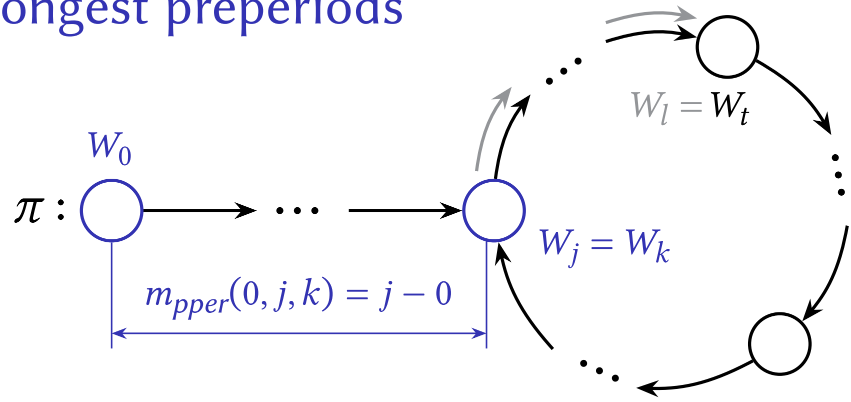
Longest periods



$$\mathscr{F}_{per}(\text{size}, \mathbf{n}) \;=\; (\varphi_{rpt}, m^2_{len}, \mathbb{Z}, \text{size}, \mathbf{n})$$

$$\varphi_{rpt}(\pi, i, j) \;=\; \begin{cases} 1, & i < j \leq l \text{ and } W_i = W_j, \\ 0, & \text{otherwise.} \end{cases}$$

Longest preperiods



$$\mathscr{F}_{pper}(\text{size}, \mathbf{n}) = (\varphi_{pper}, m_{pper}, \mathbb{Z}, \text{size}, \mathbf{n})$$

$$\varphi_{pper}(\pi, i, j, k) = \begin{cases} 1, & i = 0 < j < k \text{ and } W_j = W_k, \\ 0, & \text{otherwise.} \end{cases}$$

$\mathcal{F}_{pper}$ breakdown

the preperiod — the period

1. Find the shortest time to a repetition of states:

$$\mu_{\mathcal{F}_{pper}}(\mathscr{A}, C_0) = \min_{\gamma \in \sigma(C_0)} \{ m_{pper}(\lambda) \mid \lambda \in \varphi_{pper}(\pi_\gamma) \}$$

2. Find C_0 yielding the longest preperiod:

$$\mu_{\mathcal{F}_{pper}}(\mathscr{A}) = \max_{C_0 \subseteq S_{\mathscr{A}}} \{ \mu_{\mathcal{F}_{pper}}(\mathscr{A}, C_0) \}$$

3. Find $\mathscr{A}$ showing the largest longest preperiod:

$$\mu_{\mathcal{F}_{pper}} = \max_{\mathscr{A} \in \mathscr{RS}(\mathbf{n})} \mu_{\mathcal{F}_{pper}}(\mathscr{A})$$

Some further variants

1. Use $\mathrm{res}(W_j) = W_j$ instead of $W_j = \emptyset$.
   - adult halting
2. Use $|W_i \cap W_j| \neq \emptyset$ instead of $W_i = W_j$.
   - partial equality
3. Observe a local property $p : 2^S \to \{0, 1\}$:

$$\varphi_{loc}(p, \pi, i, j) = \begin{cases} 1, & i < j,\ p(i) = 1 \text{ and } p(j) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathrm{size}(\mathscr{A}) = |S_{\mathscr{A}}|$$

t-bit cyclic binary counter a classic

$S_{01} = \{1_i \mid 0 \le i < t\}$ $\qquad (1101)_2 \mapsto \{1_3, 1_2, 1_0\}$

$(R, \quad I, \quad P)$

$keep_{ij} : (\{1_i\}, \{1_j\}, \{1_i\})$ $\qquad t > i > j \ge 0$

keep 1_i if 1_j is set

. . . . 1 . . . 1 . . . ← least significant bit

position i position j

$set_i : (\{1_j \mid 0 \le j < i\}, \{1_i\}, \{1_i\})$ $\qquad t > i \ge 0$

set 1_i if all 1_j are set

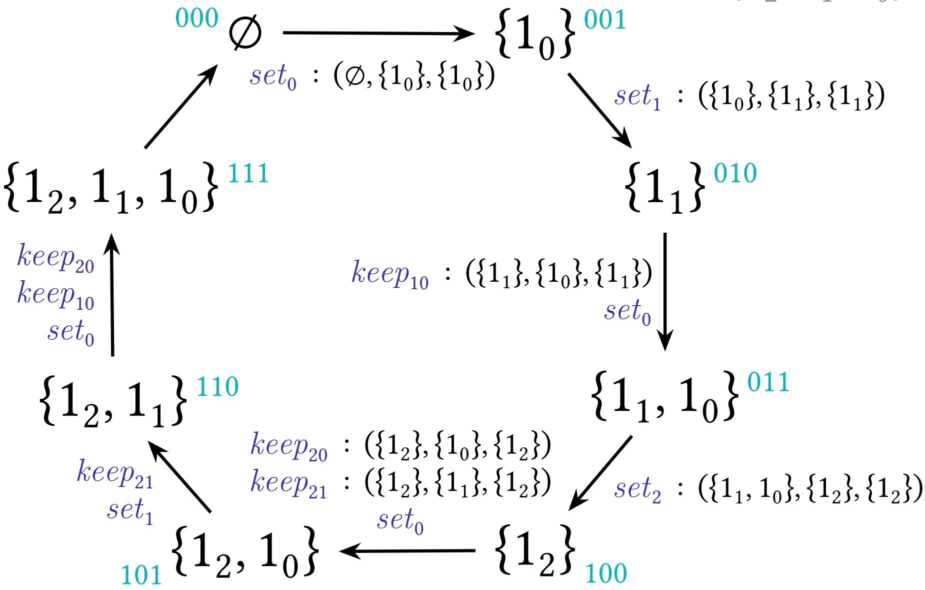
. . . . 0 1 1 . . . 1

position i

$\mathscr{A}_{01} = (S_{01}, \{keep_{ij}, set_i\})$

3-bit binary counter

$S = \{1_2, 1_1, 1_0\}$



Longest periods

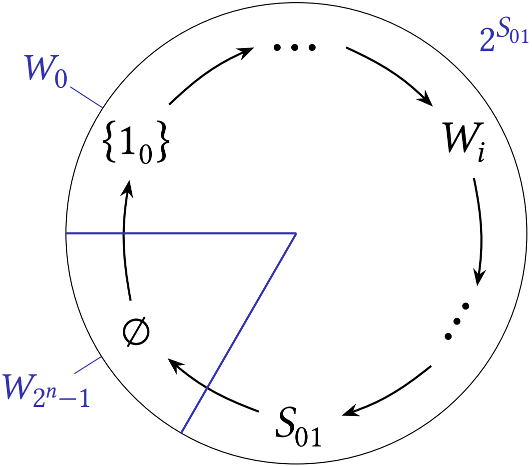
$n = \mathrm{size}(\mathscr{A}) = |S_{\mathscr{A}}|$

$\mathscr{A}_{01}$ has one period traversing the entire $2^{S_{\mathscr{A}}}$

$$\Downarrow$$

$\mathscr{A}_{01}$ has the longest period of length 2^n

Longest runs to ∅ $\quad n = \text{size}(\mathscr{A}) = |S_{\mathscr{A}}|$



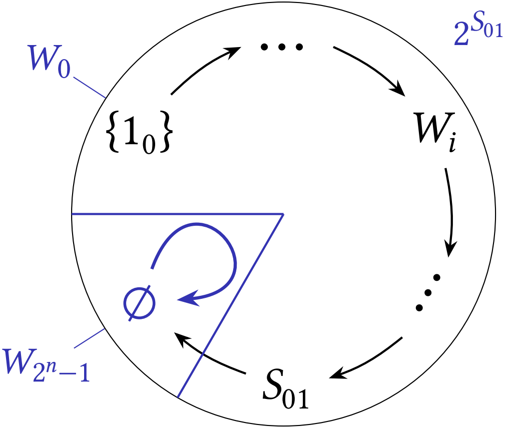
$\mathscr{A}_{01}$ has the longest run to ∅ of length $2^n - 1$.

Longest preperiods

$n = \text{size}(\mathscr{A}) = |S_{\mathscr{A}}|$

Turn $\varnothing$ into a fixed point:

$$\text{set}_0 : (\varnothing, \{1_0\}, \{1_0\}) \longmapsto \text{set}_0^x : (\{x\}, \{1_0\}, \{1_0\}), \ x \in S_{01}$$



$|\text{period}| \geq 1$

$\mathscr{A}'_{01}$ has the longest preperiod of length $2^n - 1$.

$\mathscr{A}_{enum}$: explicit traversal alternative

Order the subsets of S: $W_1, W_2, \dots, W_{2^{|S|}}$

$$(W_i, S \setminus W_i, W_{i+1}) \qquad 1 \leq i < 2^{|S|}$$

$$(W_{2^{|S|}}, S \setminus W_{2^{|S|}}, W_1)$$

$\mathscr{A}_{enum}$ has the longest period of length $2^{|S|}$.

Number of reactions used for traversing all $2^{|S|}$:

$\mathscr{A}_{enum}$	$2^{|S|}$
$\mathscr{A}_{01}$	$O(|S|^2)$

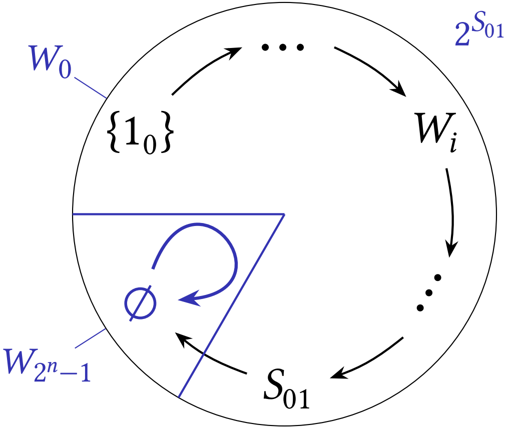
$$\text{size}(\mathscr{A}) = |S_{\mathscr{A}}|$$

but no empty sets $\neg\emptyset$

$$(R, I, P) \; : \; R \neq \emptyset, \; I \neq \emptyset, \; P \neq \emptyset$$

Preperiods and runs $n = \text{size}(\mathscr{A}) = |S_{\mathscr{A}}|, \neg\emptyset$

$\mathscr{A}'_{01}$ does not have $\text{set}_0 : (\emptyset, \{1_0\}, \{1_0\})$.



$|\text{Longest preperiod}| = |\text{longest run}| = 2^n - 1.$

Periods

$n = \text{size}(\mathscr{A}) = |S_{\mathscr{A}}|, \neg\emptyset$

No empty sets $\Longrightarrow$ $\text{res}(\emptyset) = \emptyset$

no reactions enabled in $\emptyset$

No empty sets $\Longrightarrow$ $\text{res}(S_{\mathscr{A}}) = \emptyset$

all reactions inhibited in $S_{\mathscr{A}}$

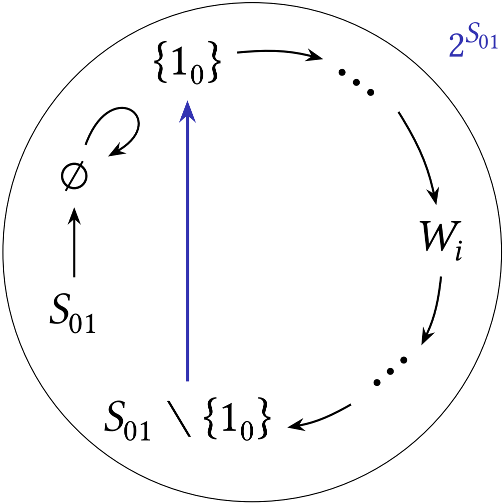
$$\Longrightarrow |\text{Longest period}_{\neg\emptyset}| \leq 2^n - 2.$$

Explicit enumeration: order $W_1, \dots, W_{2^n-1}$, $W_i \notin \{\emptyset, S\}$

$\mathscr{A}'_{enum}$: $(W_i, S \setminus W_i, W_{i+1})$ $(W_{2^{|S|}}, S \setminus W_{2^{|S|}}, W_1)$

$1 \leq i < 2^{|S|}$ $|\text{period}| = 2^n - 2$

Counter skipping $\emptyset, S_{01}$ $n = \text{size}(\mathscr{A}) = |S_{\mathscr{A}}|, \neg\emptyset$



$$|\text{Period}_{\neg\emptyset}| = 2^n - 2$$

Counter skipping $\varnothing, S_{01}$ $n = \text{size}(\mathscr{A}) = |S_{\mathscr{A}}|$, $\neg\varnothing$

$set_0^x : (\{x\}, \{1_0\}, \{1_0\})$ $\qquad x \in S_{01}$

- introduce 1_0 when 1_0 is absent

$set_i : (\{1_j \mid 0 \leq j < i\}, \{1_i\}, \{1_i\})$ $\qquad 0 < i < t$

- introduce 1_i when all less significant bits are set

$keep_{ijk} : (\{1_i\}, \{1_j, 1_k\}, \{1_i\})$ $\qquad 0 \leq j < i < t$, $0 \leq k < t, k \notin \{i, j\}$

- keep 1_i if at least one less significant bit 1_j is absent *and* one other bit 1_k is absent

$keep'_{ij} : (\{1_i, 1_0\}, \{1_j\}, \{1_i\})$ $\qquad 0 \leq j < i < t$

- keep 1_i in states *only* missing 1_j

$keep'_{ij}$ cover the cases $keep_{ijk}$ don't.

$$\mathrm{size}(\mathscr{A}) = |A|$$

number of reactions

$t = \text{size}(\mathscr{A}) = |A|$

Methods for achieving e.g. longest periods:

		$t = \lvert A \rvert$
$\mathscr{A}_{01}$	binary counter	$t(t+3)/2 = O(t^2)$
$\mathscr{A}_{enum}$	explicit enumeration	2^t

Improve $O(t^2)$?

Effective state space

The set of all states reachable by $\mathscr{A} = (S_{\mathscr{A}}, A)$:

$$\operatorname{esp}(\mathscr{A}) = \{\operatorname{res}_{\mathscr{A}}(W) \mid W \subseteq S_{\mathscr{A}}\}$$

Lemma

$$\operatorname{esp}(A) \subseteq \left\{ \bigcup_{a \in A'} P_a \mid A' \subseteq A \right\}$$

Proof. All states in $\operatorname{esp}(\mathscr{A})$ are unions of P_a. Not all possible subsets $A' \subseteq A$ can be enabled in a state $W \subseteq S_{\mathscr{A}}$. □

Size of the effective state space

Lemma

$$|\mathrm{esp}(\mathscr{A})| \leq 2^{|A|} \quad \leftarrow \text{number of reactions}$$

Proof. Follows from $\mathrm{esp}(A) \subseteq \{\bigcup_{a \in A'} P_a \mid A' \subseteq A\}$.

When $|A| \ll |S_{\mathscr{A}}|$, $|A|$ constraints $|\mathrm{esp}(\mathscr{A})|$ more strongly than $|S_{\mathscr{A}}|$.

size($\mathscr{A}$) = $|S_{\mathscr{A}}|$, size($\mathscr{A}$) = $|A|$, and ∞

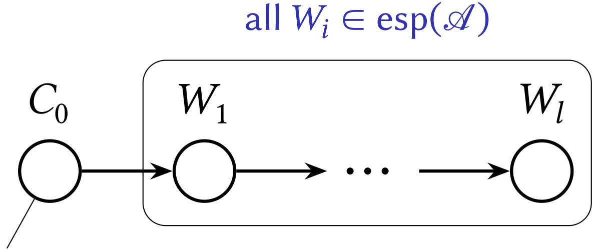
Fixed parameter	Nb. reactions	Nb. reaction systems
$\lvert S_{\mathscr{A}}\rvert$	$N_A = 2^{3\lvert S_{\mathscr{A}}\rvert}$	2^{N_A}
$\lvert A\rvert$	$N_A = \lvert A\rvert$	∞

Fixing $|A|$ does not bound $|S_{\mathscr{A}}|$

$\Rightarrow$ ∞ many reactions systems with a fixed N_A.

Longest periods

$n = \text{size}(\mathscr{A}) = |A|$



C_0 may be out of $\text{esp}(\mathscr{A})$

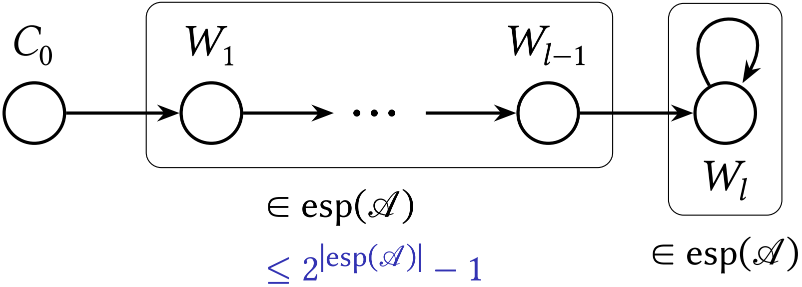
$C_0 \notin \text{esp}(\mathscr{A}) \Rightarrow C_0 \notin$ the period

$|\text{Longest period}| \leq |\text{esp}(\mathscr{A})| \leq 2^n$

Longest preperiods

$n = \text{size}(\mathscr{A}) = |A|$

The form of the longest preperiod:



$$|\text{Longest preperiod}| \leq 1 + 2^{|\text{esp}(\mathscr{A})|} - 1 \leq 2^n$$

$$|\text{Longest run to } \varnothing| \leq 2^n$$

Lower bounds? $n = \mathrm{size}(\mathscr{A}) = |A|$

Conjectures

1. Periods, preperiods, runs of length l
   $\Rightarrow$ at least $\log l$ reactions.
2. Busy beaver champions are variants of the binary counter.

Conclusions + Future work

Main contributions

1. Seeded interactive processes to mimic Turing machine runs.
2. Busy beaver frames and the generalized busy beaver games.
3. Busy beaver champions ($\text{size}(\mathscr{A}) = |S_{\mathscr{A}}|$) and bounds ($\text{size}(\mathscr{A}) = |A|$).

Future work

1. Further bounds and champions
   - more precision for $\text{size}(\mathscr{A}) = |A|$
   - other sizes, e.g. measuring individual components of the reactions
   - computational complexity of identifying champions
2. Programmatic enumeration
   - inspired by bbchallenge.org
3. Origins of Life

The Meaning of Life

Busy beaver champions $\sim$ Fittest organisms?

Parallel with biology

Reaction systems are originally inspired by biochemical systems.

Everything is finite* in physics, biology, and reaction systems.

* may be debated

No halting, open interactive processes.

No structure, no space, no multiplicities

* variants exist

Very high-level image, not a model

Only a parallel: resource-consuming dynamical systems

No faithful representation of structure, processes, quantities, concentrations, etc.

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Directions of interest

1. Different competition criteria as a representation of natural selection
2. Variations of complexity
   - structural complexity: e.g., reactions
   - algorithmic/decision complexity e.g., of identifying the champions
3. Emergence of auto-catalytic loops