$\mathcal{NP}$-hardness of pure Nash equilibrium in Scheduling and Connection Games☆

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Abstract

We introduce a framework to settle the $\mathcal{NP}$-hardness of pure Nash equilibrium for some games. The technique is simple: first, we construct a gadget without a desired property and then embed it to a larger game which encodes a $\mathcal{NP}$-hard problem in order to prove the complexity of the desired property in a game. This technique is very efficient in proving $\mathcal{NP}$-hardness of the existence of Nash equilibria. In the paper, we illustrate the efficiency of the technique in proving the $\mathcal{NP}$-hardness of the existence of pure Nash equilibria in Matrix Scheduling Games and Weighted Connection Games. Moreover, using the technique, we can settle the complexity not only of the existence of equilibrium but also of the existence of good cost-sharing protocol.

Keywords: Pure Nash equilibria, $\mathcal{NP}$-hardness, Scheduling Games, Connection Games.

1. Introduction

Equilibrium is a key concept in Game Theory. As optimization problems seek to optimal solutions, a game looks for an equilibrium. Given a game with strategy sets for players, a *pure Nash equilibrium* is a strategy profile in which each player deterministically plays her chosen strategy and no one has

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an incentive to unilaterally change her strategy. A *mixed Nash equilibrium* is similar to the pure one except that now players can pick a randomized strategy – a probability distribution over their strategy sets. In 1951, Nash proved that every game with a finite number of players, each having a finite set of strategies, always possesses a mixed Nash equilibrium. However, no similar result exists for pure Nash equilibrium.

In atomic view, pure equilibria play crucial role. For instance, on a daily day, one need to choose deterministically a route to go to work instead of choosing a route with a probability and the others with other probabilities because she cannot go, say, 1/2 on a route and 1/2 on the other route. However, in contrast to mixed equilibrium, the existence of the pure ones is not an universal property of finite games. Hence, it is important for the game designer as well as for game players to know whether a given game admits a pure equilibrium.

In the paper, we are interested in the complexity of some properties of pure Nash equilibria. Until now there are two methods, in general, to prove the $\mathcal{NP}$-hardness of a problem: using gadgets or using the PCP Theorem. We represent a technique based on the former, specifically on the gadgets called *negated* and polynomial-time reductions. The technique is the following. First, find a *negated* gadget which does not possess the desired property. (In fact, a negated gadget is a counter-example of the property.) Next, construct a family of games which encode a $\mathcal{NP}$-hard problem, and embed the gadget into. We argue that the game has the desired property if and only if there is a solution for the instance of $\mathcal{NP}$-hard problem by using the gadget to enforce rational behaviors of players in such a way that the game possesses the desired property.

Our contributions. In this paper, we present the technique as a framework and illustrate its application in different contexts. This technique is successfully applied in settling the complexity on the existence of pure Nash equilibrium in many games, such as the Matrix Scheduling Games, the Weighted Connection Games. Interestingly, this technique is not only applied to the existence of equilibrium but also to other properties such as good cost-sharing mechanism in Connection Games.

Related work. Rosenthal [13], Monderer and Shapley [11] introduced potential games which always possess a pure Nash equilibrium, for example: Congestion Games [11], Connection Games [12, chapter 19]. In these games, the existence of pure Nash equilibrium is proved by using a potential-function argument. The complexity of finding a pure equilibrium of Congestion Games is settled in [9]. Dunkel and Schulz [6], Dürr and Thang [7] showed that it is $\mathcal{NP}$-hard to decide if there exists a pure Nash equilibirium in Weighted Congestion Games and Voronoi Games, respectively.

Organization. In Section 2, we introduce the Matrix Scheduling Games and prove the complexity of the existence of pure Nash equilibrium in this game. In Section 3, we prove the complexity of the existence of pure equilibrium in Weighted Connection Games – that answers a question in [3]. Moreover, in Connection Games we show the intractability of finding a fair cost-sharing mechanism which induces an efficient equilibrium.

2. Matrix Scheduling Games

In scheduling or planning problems, there are a set of jobs (tasks) and one need to schedule jobs in order to optimize an objective function such that the schedule satisfies different constraints of jobs, for example: the constraints on the release times and deadlines, the precedence constraints and so on. Particularly, jobs need to be scheduled and completed by using different set of resources, machines. For instance, in order to make an product, a company need to use different machines that specifically produce sub-items for the final product.

We consider the game version called Matrix Scheduling Games. In the game, there are m machines, n players and a load matrix $(p_{ij})_{n \times m}$ where $p_{ij} \geq 0 \; \forall i, j$. Each player has a set of jobs that need to be executed. Players can complete their jobs by choosing a subset of machines on which their jobs will be executed, i.e., the strategy set $\mathcal{S}_i$ of player i is a collection of subsets of machines. p_{ij} represents the load contribution of player i to machine j if she uses this machine. Given a strategy profile $s = (s_1, s_2, \ldots, s_n) \in (\mathcal{S}_1 \times \mathcal{S}_2 \times \ldots \times \mathcal{S}_n)$, the *load* ℓ_j of a machine j depends on the set of jobs executed on the machine and is defined as:

$$\ell_j := \sum_{i:j \in s_i} p_{ij}.$$

The *cost* of a player is the sum of all loads of machines that the player uses:

$$c_i := \sum_{j \in S_i} \ell_j.$$

Players are selfish and they choose a strategy which minimize their cost. Remark that, without loss of generality, the strategy set of a player is *inclusion-free*, i.e., no player i possesses two strategies s_i and s'_i such that $s_i \subset s'_i$ since otherwise the player always prefer use s_i to s'_i in order to get a smaller cost.

If players' strategies are restricted to singleton machines then the game becomes the well-studied Load Balancing Games. There always exists pure Nash equilibrium on that game [8].

Proposition 1 ([8]). *The game always admits a pure Nash equilibrium if all players' strategies are singleton machines.*

Proof. We give the proof here for completeness. Given a strategy profile s, the cost of player i is the load of machine s_i on which i execute her job, i.e. $c_i = \ell_j$ where $j = s_i$. Consider the dominant potential function:

$$\Phi(s) := (\ell_1, n_1, \ell_2, n_2, \ldots, \ell_m, n_m)$$

where we rename machines in such that their loads are in decreasing order, i.e. $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_m$ and n_j is the number of machines that have the same load as machine j in strategy profile s.

If a player i has an incentive to change from machine j to machine j', meaning $\ell_j > \ell_{j'} + p_{ij}$, then $j' > j$ by definition of the indices after renaming. The new potential function after the move of i is:

$$\Phi(s') := (\ell'_1, n'_1, \ell'_2, n'_2, \ldots, \ell'_m, n'_m)$$

where $\ell_t = \ell'_t$ and $n_t = n'_t$ for all $t < j$ since there is no change on these machines.

We have that ℓ'_j is bounded above by $\max\{\ell_{j+1}, \ell_{j'} + p_{ij}, \ell_j - p_{ij}\}$. Note that $\ell_{j'} + p_{ij} < \ell_j$. If $\ell_j > \ell_{j+1}$ then $\ell'_j < \ell_j$. Otherwise, we have $\ell_j = \ell_{j+1} = \ell'_j$ but $n'_j = n_j - 1 < n_j$. Hence, the potential function strictly lexicographically decreases, hence follows the claim. □

However, without assumption on players' strategy sets, the game does not necessarily possess an equilibrium.

Fact 2. *There exists a matrix scheduling game in which there is no Nash equilibrium.*

Proof. We describe the game with no Nash equilibrium. There are 4 machines and 3 players, each one has two strategies. The strategy sets of players $1, 2$ and 3 are $\mathcal{S}_1 = \{s_1^1 = \{1, 3\}; s_1^2 = \{4\}\}, \mathcal{S}_2 = \{s_2^1 = \{1\}; s_2^2 = \{2\}\}$ and $\mathcal{S}_3 = \{s_3^1 = \{2\}; s_3^2 = \{3\}\}$, respectively. The load matrix is given as in Figure 1.

	machine 1	machine 2	machine 3	machine 4
player 1	2	∞	10	15
player 2	5	4	∞	∞
player 3	∞	4	1	∞

Figure 1: A matrix scheduling game with no Nash equilibrium.

We prove that there is no Nash equilibrium in the game by verifying all 2^3 strategy profils. In Figure 2, the first three columns represent the strategies chosen by the players. The last column shows which player is unhappy and how she can decrease her cost. For example, the first row represents a strategy profile in which all players choose their first-strategy, player 1 has cost 17 and she has an incentive to change to her second-strategy, which induces a cost 15.

□

player 1	player 2	player 3	Best reponse
s_1^1	s_2^1	s_3^1	$1 : s_1^1 \to s_1^2$ $(17 \to 15)$
s_1^1	s_2^1	s_3^2	$3 : s_3^2 \to s_3^1$ $(11 \to 4)$
s_1^1	s_2^2	s_3^1	$2 : s_2^2 \to s_2^1$ $(8 \to 7)$
s_1^1	s_2^2	s_3^2	$3 : s_3^2 \to s_3^1$ $(11 \to 8)$
s_1^2	s_2^1	s_3^1	$3 : s_3^1 \to s_3^2$ $(4 \to 1)$
s_1^2	s_2^1	s_3^2	$2 : s_2^1 \to s_2^2$ $(5 \to 4)$
s_1^2	s_2^2	s_3^1	$3 : s_3^1 \to s_3^2$ $(8 \to 1)$
s_1^2	s_2^2	s_3^2	$1 : s_1^2 \to s_1^1$ $(15 \to 13)$

Figure 2: There is no Nash equilibrium.

Using the game from previous proof as a gadget, we prove the following theorem.

Theorem 3. *Deciding whether there exists a Nash equilibrium for a given matrix scheduling game is strongly (unary) $\mathcal{NP}$-hard.*

Proof. We prove binary $\mathcal{NP}$-hardness of the decision problem with a constant number of machines by a reduction from PARTITION [10]. To show unary $\mathcal{NP}$-hardness with an arbitrary number of machines, a reduction from 3-PARTION is used. Since the proofs are basically the same, in the following, we present the reduction of the former.

In PARTITION, we are given n positive integers $a_1, \ldots, a_n$ and the question is whether there exists a partition of these n numbers into two subsets (P_1, P_2) such that the sums of elements in these subsets are equal.

We construct a matrix scheduling game in which the existence of a Nash equilibrium is equivalent to the existence of a solution for an instance of PARTITION. Given n integers $a_1, \ldots, a_n$, let B be an integer such that $\sum_{t=1}^{n} a_t = 2B$. The reduction game consists of $n + 6$ players. The first n players encode the PARTITION problem, the last three players encode the gadget of Fact 2 and the remaining three players acts as a connection between these two groups. Each of the first n players has two strategies: machine $\{1\}$ and machine $\{2\}$, the loads of job of player i ($1 \leq i \leq n$) on machines 1 and 2 are the same and equal to a_i. Player $n+1$ has two strategies: machine $\{2\}$ and machine $\{3\}$ with loads 0 and $B/2+\epsilon$, respectively; player $n+2$ has two strategies: machine $\{1\}$ and machine $\{3\}$ with loads 0 and $B/2+\epsilon$, respectively. Player $n+3$ has two strategies: machines $\{3,4\}$ and machine $\{8\}$ with loads as shown in Table 2. The last three players represent the gadget of Fact 2. Note that the load contribution of a player to a machine is infinity if it is not explicitly given.

player \ machine	1	2	3	4	5	6	7	8
1	a_1	a_1						
2	a_2	a_2						
$\vdots$	$\vdots$	$\vdots$						
$n-1$	a_{n-1}	a_{n-1}						
n	a_n	a_n						
$n+1$		0	$B/2+\epsilon$					
$n+2$	0		$B/2+\epsilon$					
$n+3$			$B/2$	3				$B/2+4$
$n+4$				2		10	15	
$n+5$				5	4			
$n+6$					4	1		

Table 1: The load matrix of the reduction game. ($p_{ij} = \infty$ if it is not explicitly given for all i,j.)

Fix an instance of PARTITION, we show that if there is a solution (P_1, P_2) for the instance then there is a Nash equilibrium for this game. Consider a strategy profile in which player i ($1 \leq i \leq n$) uses machine b for $b \in \{1,2\}$ such that $i \in P_b$, player $n+1$ uses machine 2, player $n+2$ uses machine 1, player $n+3$ uses machines $\{3,4\}$ and three others play their second-strategy (defined in Fact 2). It is straightforward that to verify that no one has an incentive to change her current strategy, hence this strategy profile is a Nash equilibrium.

In this equilibrium, all n first players' costs are B.

Inversely, suppose that there is a Nash equilibrium for this game. In this equilibrium, player $n+3$ must use machines $\{3,4\}$ since otherwise, by Fact 2, the strategy profile is not an equilibrium as it contains an unstable sub-game by the last three players. Hence, both players $n+1$ and $n+2$ play their first strategy since otherwise player $n+3$ will move. Therefore the costs of $n+1$ and $n+2$ are at most B. Moreover, $\sum_{t=1}^{n} a_t = 2B$ thus their costs are exactly B. In other words, all n first players form a partition such that the sum in two subsets are the same and therefore we obtain a solution for the PARTITION instance. □

3. Connection Games

Consider transportation companies which will open new service to connect a city to another. Theses cities may be different for different companies. They need to build a new roads on a given plan of possible roads that can be constructed in the country. The goal of each company is to open its service and minimize its cost of the construction. Similarly, network providers need to buy or rent existing optic fibers to satisfy the clients' demands of communication between two locations. They do it together but each one takes a decision to get a cost as small as possible.

Connection Game models such situation. In the game, we are given a directed graph $G = (V, E)$ with nonnegative edge costs c_e for all edges $e \in E$. There are n players, each player i has a specified source node s_i and sink t_i. Player i's goal is to build a network together with other players in order to connect her terminals s_i and t_i while paying as little as possible to do so. A strategy of player i is a path P_i from s_i to t_i in G. Given a strategy P_i for player i, the constructed network is $\cup_i P_i$, which induces the *social cost* $\sum_{e \in \cup_i P_i} c_e$ that is fully paid by players in the game.

A *cost-sharing mechanism* can be viewed as the underlying mechanism that determines how much a network serving several participants will cost to each of them. Specifically, if player i chooses a path P_i then it will be charged a cost c_i as a function of all the paths $P_1, P_2, \ldots, P_n$. The goal of a cost-sharing mechanism is to lead the game to efficient equilibria. Although there are in principle many possible cost-sharing mechanisms, research in this area has converged on a few mechanisms with good theoretical and empirical behavior in which, the most natural one is the *Shapley cost-sharing mechanism*.

Shapley cost-sharing mechanism splits the cost of an edge evenly among all players using this edge. Formally, given a strategy profile S, if n_e denotes the number of players whose path contains edge e then a cost c_e/n_e is assigned to each player using e. Thus, the total cost of player i in strategy profile S is:

$$c_i(S) := \sum_{e \in P_i} c_e/n_e.$$

The Connection Game using the Shapley cost-sharing mechanism is well studied in [1] (see also [12, chapter 19]). This game always possesses a Nash equilibrium. The inefficiency of the constructed network is measured by the price of anarchy and the price of stability. In the section, we concentrate on the price of stability. The *price of stability (PoS)* is defined as the ratio between the *best* objective value of an equilibrium of the game and that of an optimal outcome. The game with price of stability close to 1 means there is at least a "good" equilibrium. This simple argument has an important interpretation in many games, such as network games: if we envision the outcome as being initially designed by a central authority for subsequent use by selfish players, then the best equilibrium, which is close to the optimum, is an obvious solution to propose.

3.1. Weighted Connection Games

The Weighted Connection Game is a generalization of the Connection Games. In the former, each player i has additionally a weight w_i and she needs to carry the weight from her source to her sink. Consider the *weighted Shapley cost-sharing mechanism* that splits the cost of an edge proportionally to the players' weight. Formally, given a strategy profile S, let W_e be the total weight of players whose path contains e, i.e., $W_e = \sum_{i:e \in P_i} w_i$, the cost of player i on edge e

is $c_e \cdot w_i/W_e$. The total cost of player i in strategy profile S is:

$$c_i(S) = \sum_{e:e\in P_i} c_e \cdot w_i/W_e.$$

As showed in [4], there does not necessarily exist an equilibrium for Weighted Connection Games. We use the counterexample from [4] as the gadget to prove the complexity of the existence of Nash equilibria in the game. For completeness, we present here the gadget.

Lemma 4 ([4])**.** *There is a 3-player weighted connection game using the weighted Shapley cost-sharing mechanism that admits no Nash equilibrium.*

Proof. Let $w > 1$ be an arbitrary number and $\epsilon > 0$ be much smaller than $1/w^3$. Consider the network in Figure 3 with players 1, 2 and 3 who have weight $w^2, 1$ and w and need to connect their sources s_1, s_2, s_3 to the same sink t, respectively. The costs of edges are given in Figure 4 and are chosen in such a way that they satisfy the following inequalities.

$$c_5 \cdot \frac{w^2}{w^2+1} + c_9 \cdot \frac{w^2}{w^2+1} > c_7 > c_5 + c_9 \cdot \frac{w^2}{w^2+w+1} \qquad (1)$$

$$c_6 + c_9 \cdot \frac{w}{w^2+w+1} > c_8 > c_6 \cdot \frac{w}{w+1} + c_9 \cdot \frac{w}{w+1} \qquad (2)$$

If the second player uses path $e_2 \rightarrow e_5 \rightarrow e_9$ then the third player will use path e_8 (by the first half of the inequality (2)) and the first player strategy will be e_7 (by the first half of the inequality (1)). Hence, the second player will switch to the path $e_3 \rightarrow e_6 \rightarrow e_9$ to decrease her cost.

If the second player uses path $e_3 \rightarrow e_6 \rightarrow e_9$ then the third player will use path $e_4 \rightarrow e_6 \rightarrow e_9$ (by the second half of the inequality (2)) and so the first player strategy will be $e_1 \rightarrow e_5 \rightarrow e_9$ (by the second half of the inequality (1)). But in this case, the second player will switch to the path $e_2 \rightarrow e_5 \rightarrow e_9$ and get better off.

The two cases conclude the lemma.

□

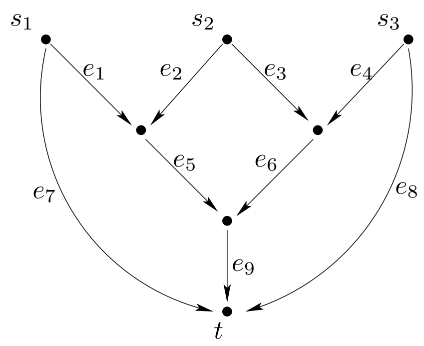

Figure 3: A 3-players weighted Shapley connection game with no Nash equilibrium.

Edge	Cost
e_1	0
e_2	3ϵ
e_3	0
e_4	0
e_5	$w^3/(w^2+w+1)-\epsilon$
e_6	$w^3/(w^2+w+1)+\epsilon$
e_7	$(w^3+w^2)/(w^2+w+1)-$ $-\ \epsilon(2w^2+1)/(2w^2+2)$
e_8	$(w^3+w^2)/(w^2+w+1)-$ $-\ \epsilon(2w+1)/(2w+2)$
e_9	1

Figure 4: The cost of edges in Lemma 4

We use the network in the previous lemma as the gadget to prove the $\mathcal{NP}$-hardness of Nash equilibrium. We construct a larger weighted connection game based on a $\mathcal{NP}$-hard problem and then embed the gadget into such that the existence of a solution for an instance of the $\mathcal{NP}$-hard problem is equivalent to the existence of Nash equilibrium in the game. Part of our constructed network is inspired by the construction of Dunkel and Schulz [6] where they proved $\mathcal{NP}$-hardness of the existence of equilibria in weighted congestion game.

Theorem 5. *It is $\mathcal{NP}$-complete to decide whether a given weighted Shapley connection game admits a Nash equilibrium.*

Proof. First, we show how to verify whether a strategy profile is an equilibrium of the game can be done in polynomial time. Consider a player i while the strategies of the others are fixed. For each edge e in graph $G(V,E)$, we can compute the cost c'_e that player i has to pay on this edge if she uses it. Then, we can construct a graph $G'(V,E)$ which is the same as G except that now the cost of each edge e is c'_e. On graph G', we can compute the shortest path from s_i to t_i in polynomial time and verify if player i is happy in the given strategy profile in graph G. Hence, applying the procedure for all players, we can answer the question in polynomial time.

In the following, we prove the $\mathcal{NP}$-hardness by a reduction from Monotone3Sat[1] [10]. Consider an instance of Monotone3Sat with variable set $X = \{x_1, \ldots, x_n\}$ and clause set $C = \{c_1, \ldots, c_m\}$. Each clause contains at most three literals and either all literals in a clause are negated or all are unnegated. Deciding whether there is a satisfying assignment for this instance is $\mathcal{NP}$-hard.

We construct a game such that for each literal $x \in X$, there is a *literal player* p_x with weight 1, source x and sink $\bar{x}$. Moreover, each clause $c \in C$ gives rise to a *clause player* p_c with weight 1, source c and sink $\bar{c}$. Besides, we have three additional players p_1, p_2, p_3 of weight $w^2, 1, w$ and terminals $(s_1, t), (s_2, t), (s_3, t)$, respectively. These three players will play the role of the gadget from Lemma 4. One additional player p_4 has weight 1 and terminals (s_4, t_4). Remark that in the reduction network, all players have weight 1 except players p_1 and p_3. In our construction, ϵ is positive and arbitrarily small.

We first describe the sub-network for all players p_x and p_c, $x \in X, c \in C$. Part of this sub-network is illustrated in Figure 5. For player p_x, there are two paths P_x^0, P_x^1 from x to $\bar{x}$. Let $n_x := |\{c \in C | x \in c\}|$ and $n_{\bar{x}} := |\{c \in C | \bar{x} \in c\}|$. Path P_x^1 consists of $(2n_x + 2)$ edges and path P_x^1 consists of $(2n_{\bar{x}} + 2)$ edges. On each path, the cost of all oddth edges is 0 and that of all eventh edges is 2 except for the last edge. If $n_x > n_{\bar{x}}$ then the cost of the last edge on path P_x^0 is $(n_x - n_{\bar{x}})$ and that on path P_x^1 is 0. Otherwise, the cost of the last edge on path P_x^0 is 0 and that on path P_x^1 is $(n_{\bar{x}} - n_x)$. Each player p_c also has two paths P_c^0, P_c^1 from c to $\bar{c}$. Path P_c^0 consists of two edges with cost $9/2 + \epsilon$ and 1. Path P_c^1 consists of seven edges and is constructed for $c = c_j$ in the order $j = 1, \ldots, m$ as follows. For a positive clause $c = c_j = (x_{j_1} \vee x_{j_2} \vee x_{j_3})$ with $j_1 < j_2 < j_3$, path P_c^1 starts with the edge connecting source c to the first inner node v_1 on path $P_{x_{j_1}}^1$ that has only two incident edges so far. The second edge

[1] The choice of Monotone3Sat is driven by the simplicity in drawing the network. We can make reduction from 3Sat with exactly the same arguments.

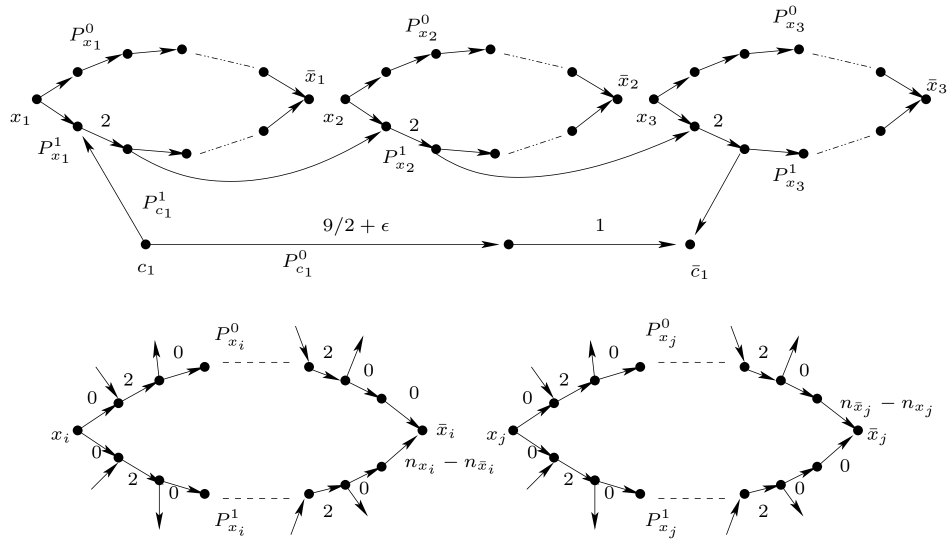

Figure 5: Network of players p_x and p_c. Two paths of player p_{c_1}, for $c_1 = x_1 \vee x_2 \vee x_3$, are illustrated.

is the unique edge (v_1, v_2) of path $P^1_{x_{j_1}}$ that has v_1 as its start vertex. The third edge connects v_2 to the first inner node v_3 on path $P^1_{x_{j_2}}$ that has only two incident edges so far. The fourth edge is the only edge (v_3, v_4) on $P^1_{x_{j_2}}$ with start vertex v_3. Similarly, the fifth edge is the edge connecting v_4 to the first inner node v_5 of $P^1_{x_{j_3}}$ which has only two incident edges so far, followed by (v_5, v_6). The last edge of P^1_c connects v_6 to sink $\bar{c}$. For a negative clause, the construction is similar. The difference is that a positive clause concerns only paths (of literal players in the clause) of superscript 1 while a negative clause concerns only those of superscript 0.

The second part of the network consists of the four players p_1, p_2, p_3 and p_4, shown in Figure 6. First three players represent a network (with edge costs) defined in Lemma 4. In addition, this network has an additional edge e_{10} of cost 0. Player p_4 has two paths P^0_4, P^1_4 connecting her source s_4 and her sink t_4. Path P^1_4 consists of edge e_8 and an additional edge (t, t_4) of cost $m - c_8/(w+1) - \epsilon$. Path P^0_4 shares edges of cost 1 with all paths $P^0_c, \forall c \in C$ and contains some additional edges (of cost 0) connecting those. Note that each player in our network possesses two strategies. We call a *0-path* (*1-path*, resp) of a player the path with superscript 0 (1, resp).

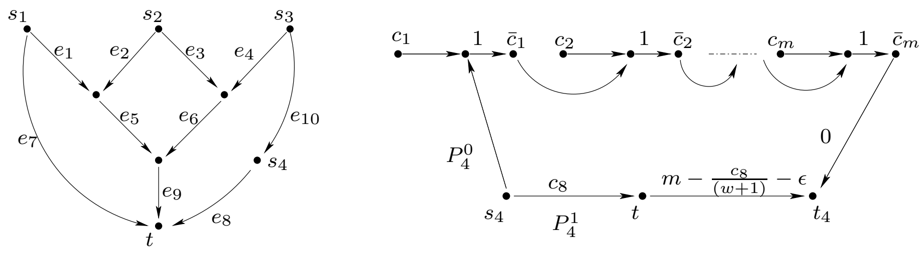

Figure 6: Network of players p_1, p_2, p_3 and p_4 (an edge has cost 0 if not explicitly given).

Given a satisfying assignment for the MONOTONE3SAT's instance, consider the following strategy profile. Players p_x ($x \in X$) use their b-path ($b \in \{0, 1\}$) corresponding to the value b of x in the solution, players p_c ($c \in C$) use their 1-path, player p_4 uses her 1-path and players p_1, p_2, p_3 use paths $(e_7), (e_3 \rightarrow e_6 \rightarrow e_9), (e_{10} \rightarrow e_8)$ respectively. We argue that this strategy profile is a Nash equilibrium. Player p_x has no incentive to switch her strategy because her cost stays the same (it equals $\max\{n_x, n_{\bar{x}}\}$) even if she changes the strategy (by the trick that we add a new edge of cost $|n_x - n_{\bar{x}}|$ to some paths). Player p_c's cost is at most 5 since in a satisfying assignment, she shares at least an edge of cost 2 with a player $p_x, x \in X$. Observing that if using 0-path, player p_c may share only one edge of cost 1 with player p_4 so if she switches to 0-path, in the best case, her cost would be $9/2 + \epsilon + 1/2 > 5$. Hence, player p_c is happy on the 1-path. The cost of player p_4 is $m - \epsilon$ and she is happy on her current strategy. It is easy to verify that all three players p_1, p_2 and p_3 are also happy.

Suppose there is a Nash equilibrium for this game. Hence, in this equilibrium, player p_4 must use her 1-path since otherwise it is not an equilibrium since there exists an unstable sub-network (by Lemma 4). The cost of p_4 is at least $m - \epsilon$ and this happens in case p_4 shares edge e_8 with p_3. Therefore all players p_c use her 1-path because if there is a player p_c using her 0-path, player

p_4 has an incentive to change her strategy and get a cost at most $m - 1/2$. The fact that players p_c ($\forall c \in C$) play their 1-path means that, for each $c \in C$, there is at least one player p_x sharing an edge with p_c (otherwise, p_c will change her strategy). Hence, the assignment, in which $x_i = 1$ if p_x uses 1-path and $x_i = 0$ otherwise, is a satisfying assignment for the MONOTONE3SAT instance. $\square$

3.2. Cost-Sharing mechanism for Good Equilibrium

In the previous subsection, we consider the Shapley cost-sharing mechanism as the allocation of the cost of the constructed network to players. Under this cost-sharing mechanism, the PoS is $\Omega(\log n)$ [1] where n is the number of players in the game. A natural question is whether there exists a cost-sharing mechanism which guarantees a small inefficiency of the game. Chen et al. [5] have studied the inefficiency of the Connection Games while considering the set of *admissible* cost-sharing mechanisms. A cost-sharing mechanism is *admissible* if it satisfies:

(i) *Budget-balance*: the cost of each edge in the constructed network is fully passed on to its users;

(ii) *Separability*: the cost shares of an edge are completely determined by the set of players that use it;

(iii) *Stability*: for every network using the cost-sharing mechanism, there is at least one Nash equilibrium.

The purpose is to design an admissible cost-sharing mechanism such that for all networks, the PoS induced from this mechanism is small, say constant.

In Connection Games where all players have the same sink, there is a cost-sharing mechanism which induces PoS = 1 for every network [2]. Nevertheless, the best Nash equilibrium is not social optimum in general case. Precisely, Chen et al. [5] obtained that for any admissible cost-sharing mechanism, the PoS is at least 3/2. Here, we present our proof for this lower bound.

Lemma 6 ([5]). *There is a directed network whose PoS is at least 3/2 for any admissible cost-sharing mechanism.*

Proof. Consider the network illustrated in Figure 7 where player p_i has source/sink (s_i, t_i) for $1 \leq i \leq n$. The edges $e_1, e_2, \ldots, e_n$ have cost $1 + 1/n$. The *backbone* path consists of edges $e_1, e_2, \ldots, e_n$ alternating with edges of cost 0. Players can use the backbone path or other paths to connect her terminals. Player p_i $(2 \leq i \leq n)$ has a path containing three edges of cost 1/2. We call *one-hop-source edge*, *backward edge* and *one-hop-sink edge* the first, the second and the last among these three edges, respectively. The optimum, where all players use the backbone path, is of cost $n + 1$. We will prove that for any admissible cost-sharing mechanism, an equilibrium of n players on this network has cost at least $3(n - 1)/2 + 1$.

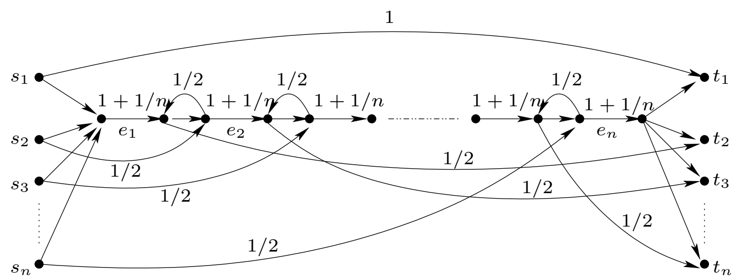

Figure 7: Network of n players whose PoS = 3/2 (an edge has cost 0 if it is not explicitly given).

Fix an admissible cost-sharing mechanism. We claim that in an equilibrium, no player entirely uses the backbone path. Suppose in an equilibrium there are $k - 1$ players entirely using the backbone path, i.e., $(n - k + 1)$ other players use at least one of their one-hop edges in order to connect their terminals. Observe that the cost restricted on the backbone path of each one in these $(n-k+1)$ players is at most 1. For the first player, if her restricted cost on the backbone is strictly larger than 1 then she will switch to the other path. For other player p_i $(2 \leq i \leq n)$, if that cost is larger than 1, she would change her path restricted on the backbone by an one-hop edge with or without a backward edge of cost $1/2$. Therefore, the total cost shared by the $(k-1)$ players is at least $n+1-(n-k+1)=k$. In other words, there is a player p_i who pays strictly more than 1 on the backbone path. So this player is not the first one. For $i \geq 2$, p_i pays strictly more than $1/2$ either on her path before e_i or after (including) e_i. Hence, this player can decrease her cost by switching her strategy. So the strategy profile is not an equilibrium — contradiction.

Consider an equilibrium S and let $R = \{j : e_j \in S\}$. If $R = \emptyset$ then in S each player p_i $(2 \leq i \leq n)$ pays $3/2$ and player p_1 pays 1. The total cost is $3/2 \cdot (n-1) + 1$. If $R \neq \emptyset$, let i be the smallest element in R. If $i > 1$ then p_i uses her one-hop-source edge and no player p_j uses an edge e_j with $j \leq i$ on the backbone by minimality of i. Thus, player p_i fully pays edge e_i. In this case, the total cost of p_i is $3/2 + 1/n$ and she has an incentive to switch her strategy. Hence $i = 1$. Let T be the set of players using e_1 in S and $k := \min\{j : p_j \in T\}$. Remark that $k > 1$ and no one uses the full backbone path. Since S is an equilibrium, edge e_{k-1} is shared by p_k and some other player of index $\ell < k$ (otherwise, p_k is the only one who pays edge e_{k-1} and she is not happy). Since $\ell < k$ and p_ℓ uses an edge e_{k-1}, player p_ℓ must use a path containing $\{e_{k-1}, e_{k+1}, \ldots, e_n\}$ (she can not use her one-hop-sink edge). The edges $\{e_1, \ldots, e_{k-1}\}$ are used by p_k. That means the backbone path is entirely bought. Moreover, no one uses entirely the backbone path, i.e., each player uses at least an edge of cost $1/2$. Hence, the total cost in S is $(n+1)+1/2\cdot(n-1)$. □

Consider the Shapley cost-sharing mechanism. This mechanism has a good property, namely fairness — it divides evenly the cost of an edge to players using the edge. We are interested in designing admissible cost-sharing mechanisms which also possess a fairness property. We define the property ϵ-fairness which is desired in a cost-sharing mechanism. Intuitively, in these mechanisms, if an edge is shared by several players, no one pays a too large fraction of the cost.

Definition 7. Given $0 < \epsilon < 1/2$, a cost-sharing mechanism is *ϵ-fair* if for any edge e, if there are at least two players sharing e then no one pays more than $1 - \epsilon$ fraction of this edge cost.

Unfortunately, the following theorem shows that it is intractable to find an ϵ-fair admissible cost-sharing mechanism which ensures small inefficiency of equilibria of the game. It also highlights an intuition in designing a cost-sharing mechanism for the Connection Games with good PoS: this kind of cost-sharing mechanism may not be fair.

Theorem 8. *Given a network $G(V,E)$ with edge costs $c : E \to \mathbb{Q}$ and a set of players, deciding whether there exists an ϵ-fair cost-sharing mechanism such that the PoS $\leq 3/2 - \delta$ is $\mathcal{NP}$-hard, where $\delta > 0$ can be chosen arbitrarily small.*

Proof. Again, we reduce from MONOTONE3SAT. Consider an instance of MONOTONE3SAT with variable set $X = \{x_1, \ldots, x_n\}$ and clause set $C = \{c_1, \ldots, c_m\}$. Let α, β, M, k be parameters satisfying the following inequalities:

$$2\alpha + (1-\epsilon)\alpha < \beta < 3\alpha \tag{3}$$

$$(m-1)\beta + (1-\epsilon)\beta < M < M+2 < m\beta \tag{4}$$

$$\frac{2}{\delta}(M + m\beta + nm\alpha) < k \tag{5}$$

We can choose, for instance, $\alpha = \frac{2}{\epsilon}, \beta = \frac{6}{\epsilon} - 1, M = m\beta - 4, k = 2(M + m\beta + nm\alpha)/\delta$.

We will create a game which has one player p_x with origin x and destination $\bar{x}$ for every literal $x \in X$ and one player p_c with origin c and destination $\bar{c}$ for every clause $c \in C$. Besides, we have k additional players who play the role of the gadget from Lemma 6. One more additional player p has terminals s, t.

The network for all players p_x and p_c for $x \in X, c \in C$ is similar to that in the proof of Theorem 5. For player p_x, there are two paths P_x^0, P_x^1 from x to $\bar{x}$. Let $n_x := |\{c \in C | x \in c\}|$ and $n_{\bar{x}} := |\{c \in C | \bar{x} \in c\}|$. Path P_x^1 consists of $(2n_x + 2)$ edges and path P_x^1 consists of $(2n_{\bar{x}} + 2)$ edges. On each path, the cost of all oddth edges is 0 and that of all eventh edges is α except the last edge. If $n_x > n_{\bar{x}}$ then the cost of the last edge on path P_x^0 is $(n_x - n_{\bar{x}})\epsilon\alpha$ and that on path P_x^1 is 0. Otherwise, the cost of the last edge on path P_x^0 is 0 and that on path P_x^1 is $(n_{\bar{x}} - n_x)\epsilon\alpha$. For each player p_c, she also has two path P_c^0, P_c^1 from c to $\bar{c}$. Path P_c^0 consists of one edge with cost β. Path P_c^1 consists of seven edges and are constructed for $c = c_j$ in the order $j = 1, \dots, m$ in the same manner as in the proof of Theorem 5.

The second part of the network consists of k players forming a gadget as described in Lemma 6. Additional player p has two path-strategies P_p^0, P_p^1 connecting her terminals s, t. Path P_p^0 consists of $2m$ edges, each eventh edge is shared with a player $p_c, c \in C$ (these edges have cost β) and other edges have cost 0. Path P_p^1 consists of $2k+1$ edges, where $2k-1$ edges are on the backbone path of the gadget, an edge of cost M connects source s to the beginning point of the backbone path and an edge of cost 0 connecting the endpoint of the backbone path to sink t. Again, we call a *0-path* (*1-path*, resp.) of a player is her path with super index 0 (1, resp.).

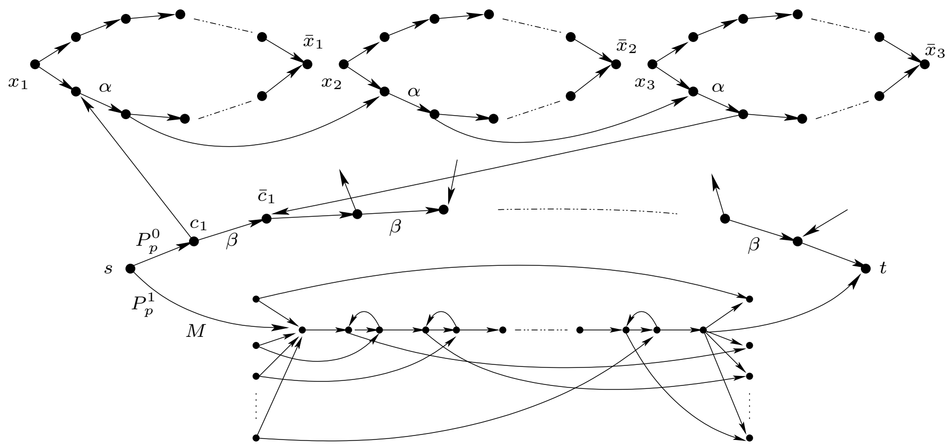

Figure 8: A part of network with player p, players on the gadget and players $p_{c_1}, p_{x_1}, p_{x_2}, p_{x_3}$ where $c_1 = (x_1 \vee x_2 \vee x_3)$.

Given a satisfying truthful assignment, we argue that the following strategy profile together with a cost-sharing mechanism is a Nash equilibrium and the cost of this equilibrium is within a factor 3/2 of OPT. Consider a strategy profile in which players p_x ($x \in X$) use their b-path ($b \in \{0, 1\}$) corresponding to the value b of x in the truthful assignment, players p_c ($c \in C$) use their 1-path, player p uses her 1-path and players in the gadget use the backbone path.

Let ξ be a cost-sharing mechanism such that

(i) if a player $p_x,\ x \in X$ and a player $p_c,\ c \in C$ share an edge then p_c pays an $(1-\epsilon)$ fraction of the edge, the rest is paid by p_x;

(ii) in the gadget, player p_i $(2 \le i \le n)$ pays $1/2$ on edge e_{i-1} and $1/2$ on edge e_i, player p_1 pays 0 and player p pays $1/2 + 1/n$ on edges e_1, e_n and she pays $1/n$ on edges e_i for $2 \le i \le n-1$.

Clearly, ξ is an ϵ-fair admissible cost-sharing mechanism.

We show that everyone in this strategy profile is happy and bound the total cost. Player p_x has no incentive to switch her strategy because her cost stays the same even if she changes her path (due to the trick that we add a new edge of cost $\epsilon(n_x - n_{\bar{x}})\alpha$ or $\epsilon(n_{\bar{x}} - n_x)\alpha$ to some paths). Player p_c is happy because her current cost is at most $2\alpha + (1-\epsilon)\alpha$ which is smaller than β – the cost she would pay if she unilaterally switches the strategy. The same holds for player p, since she must pay $m\beta$ if she changes strategy instead of the cost $M + 2$. All players on the gadget are also happy. Observing that the optimum of the gadget is the backbone path of cost k and since no one in the gadget can use a path outside, the cost of OPT of the network is at least k. By the choice of parameters, value k dominates the cost of all edges outside the gadget. Therefore the cost of the Nash equilibrium described above is smaller than $3/2k$, or in the other words, it is within a factor $3/2 - \delta$ of the OPT by inequality (5).

Suppose there is a Nash equilibrium derived from an ϵ-fair cost-sharing mechanism ξ such that the cost of this equilibrium is within a factor $3/2$ of the OPT in this game. Hence, in this equilibrium, player p uses her 1-path and shares the backbone path of the gadget (otherwise, the cost of this equilibrium is at least $3/2k$ by Lemma 6 and is greater than $(3/2 - \delta)OPT$ by (5)). It means that no player $p_c, c \in C$ uses her 0-path (one-hop path of cost β) since otherwise player p will switch her strategy (by (4)). Therefore, on her 1-path, player p_c shares some edge with some player p_x (by (3)). For all $x \in X$, assign x to the value 0 or 1 depending on 0-path or 1-path that player p_x uses in the equilibrium. This assignment is satisfied because each clause c $(c \in C)$ has value 1. □

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