

On Randomized Strategy-Proof Mechanisms without Payment for Facility Location Games

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Abstract

We characterize the performance of randomized strategyproof and group-strategyproof social choice rules, for placing a facility on the nodes of a metric network inhabited by N autonomous self-interested agents. Every agent owns a set of locations and caters to minimization of his cost which is the total distance from the facility to his locations. Agents may misreport their locations, so as to manipulate the outcome. A central authority has a set of allowable locations where the facility could be opened. The authority must devise a mechanism that, given the agents reports, places the facility in an allowable location that minimizes the utilitarian social cost — the sum of agents costs. We consider strategyproof and group strategyproof such mechanisms. A mechanism is strategyproof if no agent may misreport his locations and be better off; it is group-strategyproof if no coalition of agents may jointly misreport their locations and all agents in the coalition decrease their distance from the facility location selected by the mechanism. The requirement for (group-)strategyproofness in this setting makes optimum placement of the facility impossible and, therefore, we consider approximation (group-) strategyproof mechanisms.

We study strategyproof and group strategyproof mechanisms. For the former, we give a simple 3-approximation mechanism that is randomized over a set of preferential locations of agents. We also provide an asymptotic lower bound of 2 for randomized strategyproof mechanisms. For the latter, a deterministic group strategyproof mechanism which is inspired from the dictatorship mechanism is $(2N + 1)$ -approximation. Even the simplicity of that mechanism, we showed that it is asymptotically optimal up to a constant. Our $\Omega(N^{1-\epsilon})$ lower bound that randomization cannot improve over the approximation factor achieved by the deterministic mechanism, when group-strategyproofness is required.

1 Introduction

In a metric space inhabited by N agents, we consider the problem of using agents' reports for their positions to select a facility location in order to minimize aggregatively the agents' distances from their locations to the facility. The rule by which the facility location are selected is public knowledge. Each agent owns a set of locations. Agent are self-interested and each one aims at minimizing his individual cost, i.e. total distance from his locations to the facility. To this end, he may manipulate the outcome of the facility location rule, by strategically misreporting his position. We study the power and limitations of *strategyproof* or even *group-strategyproof mechanisms*, that approximate the optimum aggregate cost over all agents within a bounded factor. In the paper, we consider the aggregate cost function as the *utilitarian* social cost, i.e. the *sum* of costs of all agents. A strategyproof (SP) mechanism ensures that no agent may misreport his positions and be strictly better off. A group-strategyproof (GSP) mechanism is resilient to coalitional misreports. In effect, the mechanism constitutes a rule for placing the

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facilities, that renders truthful report of the agents' positions a dominant strategy for each agent regardless of the other agents' actions.

For an application of the problem model, consider installation of a public service facility such as a hospital or a library within the region of a municipality. The authorities announce the rule (mechanism) by which a location will be chosen within the region and run a survey over the population; each inhabitant may declare the spot in the region that she prefers facility installation at. Every inhabitant wishes to minimize her individual distance from a facility location spot, possibly by misreporting her preference to the authorities. If the mechanism is (G)SP, it is to the best interest of everyone to reveal the exact coordinations of their place of residence. The setting has many other applications on computer, telecommunication networks where network locations may be easily manipulated by reporting false IP address, communication routers, etc.

The problem of (G)SP optimization of the aggregate cost over all agents was studied first by Procaccia and Tennenholtz [10] and received considerable attention thereafter [5, 1, 6]. It falls within the broader context of *mechanism design without payments*, i.e. designing social choice rules over the preferences of the agents, without eliciting payments from the agents to incentivize them. The study of algorithmic concerns in mechanism design has received an extensive attention in the recent literature, since the seminal work of Nisan and Ronen [9]. Several works have appeared concerning the algorithmic design of (G)SP mechanisms that involve some — even computationally hard — optimization problem over (and for) a set of agents, based on truthful reports of private information from the agents. The algorithmic hardness may then be inherent in the optimization problem (**NP**-hardness). A crucial aspect in most of the settings that have been studied is facilitation of incentive compatibility by elicitation of appropriate *payments* from the side of the agents to the mechanism deployer. An example is the VCG mechanism, which may use a — possibly exponential time — algorithm to solve optimally a problem such as the facility location we consider, and then elicit payments from the players. It is in the optimality of the solution and in the way the payments are defined, that VCG ensures strategyproofness. However, in many social choice setting, monetary transfer may be inappropriate, even impossible, and computationally difficult due to legal or ethical reasons [12, 10]. Hence, the issue here are twofold: the computational intractability and the absence of payment.

In the context the optimization problem that we consider is computationally tractable. We study how the demand for (group-) strategyproofness without payment limits the optimization prospects of the designed mechanisms and therefore leads to approximation upper and lower bounds. The celebrated Gibbard-Shatterworth impossibility theorem [4, 11] states that strategyproofness cannot be achieved for arbitrary preferences of the agents. Moulin [8] considered *single-peaked* preferences in the line metric, i.e. the case where the metric space is a line and each agent has (exactly) one preferred to any other location. In the paper, we consider a broader context of general metric spaces and each agent may have multiple preferential locations (although all such locations induce the same cost for the agent).

Contribution Our two main results in this paper concern performance characterization of randomized SP and GSP mechanisms for placing a single facility on arbitrary metric graphs. For randomized SP mechanisms, we give a simple 3-approximation mechanism that is randomized over a set of preferential locations of agents. We also provide an asymptotic lower bound of 2 for any randomized strategyproof mechanism. For GSP mechanisms, a deterministic group strategyproof mechanism which is inspired from the dictatorship mechanism is devised, which is $(2N + 1)$ -approximation. Even the simplicity of that mechanism, we showed that it is asymptotically optimal up to a constant. Our $\Omega(N^{1-\epsilon})$ lower bound that randomization cannot improve over the approximation factor achieved by the deterministic mechanism, when

group-strategyproofness is required. Our result also confirms affirmatively a conjecture posed by Alon et al. [1].

1.1 Related Work

There has been some early work on characterizing strategyproof facility location mechanisms without payments on lines [8] and on circles [12]. However, Procaccia and Tennenholtz [10] initiated in the study of approximating the optimum social cost under the constraint of (group) strategyproofness. The authors considered facility location problems on metric spaces that are lines. A classical result by Schummer and Vohra [12] dictates that the only deterministic SP mechanism that one may hope for a circle is a *dictatorship* (i.e. a mechanism that fixes an agent and always opens the facility as the preference of the agent, while totally ignore the reported locations of other agents). Such a mechanism has approximation ratio $\Omega(N)$. The model in [1] is closest to ours. Alon et al. [1] considered the problem on circles and on general metric networks in which each agent own only one location and the set of allowable locations where the facility could be opened is the set of all vertices of the graph. They show that the *randomized dictatorship* mechanism that selects a location uniformly at random, is SP and approximates the social cost within a factor of $2 - \frac{2}{N}$ for any metric network. The authors conjectured that no GSP randomized mechanism may achieve $o(N)$ approximation ratio. This conjectured is settled in the present paper.

In additional to the utilitarian objective, Procaccia and Tennenholtz [10], Alon et al. [1, 2] also considered another target, the *egalitarian* social cost, which is the maximum among all agents' costs. They obtained tight bounds for deterministic, as well as randomized (group-) strategyproof mechanisms.

Procaccia and Tennenholtz [10] also initiated the study of locating two facilities. Under the egalitarian objective, the deterministic approximability was fully identified to be 2 in [10] (GSP upper bound and SP lower bound). For randomized mechanisms they proved a $\frac{5}{3}$ SP upper bound and a $\frac{3}{2} - \epsilon$ SP lower bound. Under the utilitarian objective, the authors gave a deterministic GSP $(N - 2)$ -approximation mechanism and a constant lower bound for deterministic SP mechanisms. The latter was proved to be asymptotically tight in [6]. Apart from an $\Omega(N)$ lower bound on lines, Lu et al. [6] also gave a randomized SP 4-approximation mechanism for general metric spaces. They also considered the case of a circle, for which they designed a deterministic GSP $(N - 1)$ -approximation mechanism. This bound is asymptotically the best possible, up to a factor of 2, according to the lower bound they proved.

2 Preliminaries

We consider a metric space (Ω, d) , where $d : \Omega \times \Omega \rightarrow \mathbb{R}$ is the metric function. Let \mathcal{N} be the set of N agents and each agent $i \in \mathcal{N}$ owns a set x_i of w_i locations $\{x_{i1}, \dots, x_{iw_i}\}$ where $x_{ij} \in \Omega$ for $1 \leq j \leq w_i$. A *location profile* (or *strategy profile*) is a vector $\mathbf{x} = (x_1, \dots, x_N)$ where x_i is a set of locations of agents i for $i = 1, \dots, N$. Let $\mathcal{F} \subseteq \Omega$ be a set of allowable locations where the facility can be opened. A *deterministic* mechanism is a mapping f from the set of strategy profiles to a location in \mathcal{F} . Given a reported location profile \mathbf{x} the mechanism's output is $f(\mathbf{x})$ and the individual cost of agent $i \in \mathcal{N}$ under mechanism f and profile \mathbf{x} is the total distance from his locations to the facility, denoted by $c_i(f, \mathbf{x})$:

$$c_i(f, \mathbf{x}) = \sum_{j=1}^{w_i} d(f(\mathbf{x}), x_{ij})$$

A *randomized* mechanism is a function f from the set of location profiles to $\Delta(\mathcal{F})$ where $\Delta(\mathcal{F})$ is the set of distribution over \mathcal{F} . The cost of agent i is now the expected total distance from his locations to the facility over such distribution:

$$c_i(f, \mathbf{x}) = \mathbb{E} \left[\sum_{j=1}^{w_i} d(f(\mathbf{x}), x_{ij}) \right]$$

The *social cost* of a mechanism f is the sum of individual costs of agents:

$$C(f, \mathbf{x}) = \sum_{i \in N} c_i(f, \mathbf{x})$$

We say that a mechanism f is *r-approximation* if for any profile \mathbf{x} ,

$$C(f, \mathbf{x}) \leq r \cdot OPT(\mathbf{x})$$

where $OPT(\mathbf{x})$ is the optimal social cost.

We will be concerned with *strategyproof* (SP) and *group-strategyproof* (GSP) mechanisms, which render truthful revelation of the agents' location a dominant strategy for the agents. We provide definitions for strategyproofness and group-strategyproofness below:

Definition 1. (Strategyproofness) Let \mathbf{x} denote the location profile of a set \mathcal{N} of N agents, over the metric space (Ω, d) . A mechanism f is *strategyproof* if for every agent $i \in \mathcal{N}$ and for every location profile \mathbf{x}' with $x'_j = x_j$ for $j \neq i$, $c(f(\mathbf{x}'), x_i) > c(f(\mathbf{x}), x_i)$.

Definition 2. (Group-Strategyproofness) Let \mathbf{x} denote the location profile of a set \mathcal{N} of N agents, over the metric space (Ω, d) . A mechanism f is *group-strategyproof* if for every non-empty subset of agents $\mathcal{I} \subseteq \mathcal{N}$ and for every location profile \mathbf{x}' with $x'_j = x_j$ for $j \notin \mathcal{I}$, there is an agent $i \in \mathcal{I}$ with $c(f(\mathbf{x}'), x_i) > c(f(\mathbf{x}), x_i)$.

We note that a stronger notion of group-strategyproofness has been used elsewhere, which prescribes that if any person in the coalition strictly benefits, at least one person is strictly worse off. The definition we use here has been referred to as weak group-strategyproofness that follows definition in previous work [10, 1, 6] and in the context of designing cost-sharing mechanisms [7]. Besides, a GSP mechanism is clearly SP, but the inverse does not hold in general.

Given a subset $U \subset \Omega$ in the metric space, we define $\text{med}(U)$ a location where the facility could be opened and the total distance from that location to locations in U is minimized, i.e $\text{med}(U) := \arg \min \{v \in \mathcal{F} : \sum_{u \in U} d(v, u)\}$, break tie arbitrarily. Given U , $\text{med}(U)$ can be computed in linear time. For an agent $i \in \mathcal{N}$ with reported locations x_i , we define $y_i := \text{med}(\{x_{i,1}, \dots, x_{i,w_i}\})$. Remark that for any location $v \in \Omega$, $\sum_{i=1}^N \sum_{j=1}^{w_i} d(v, x_{i,j}) \geq \sum_{i=1}^N \sum_{j=1}^{w_i} d(y_i, x_{i,j})$.

3 Randomized strategy proof mechanisms

In this section, first we present a simple randomized strategy proof mechanism which is 3-approximation. The mechanism is inspired by mechanism and ideas in [12, 1]. Then we show a lower bound of $2 - o(1)$ on the performance of randomized SP mechanisms for placing a facility on graphs.

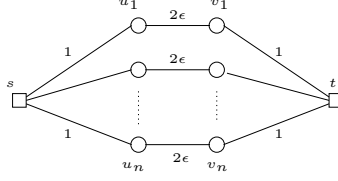


Figure 1: Network of lower bound $2 - 4N^{-1/3}$.

Randomized mechanism Given a location profile $\mathbf{x} = (x_1, \dots, x_N)$ where $x_i = \{x_{i1}, \dots, x_{iw_i}\}$ for $1 \leq i \leq N$. Let $y_i = \text{med}(x_i)$. Open the facility at y_i with probability w_i/W where $W = \sum_{i=1}^N w_i$.

Theorem 1. *The mechanism is strategy proof and that yields 3-approximation. Moreover, no randomized strategy proof mechanism has approximation ratio better than $2 - 4N^{-1/3}$ even if each agent possesses only one location.*

The theorem is proved by the following lemmas.

Lemma 1. *The mechanism is strategy proof and that yields 3-approximation.*

Proof. Suppose that an agent i lies about his locations such that y_i is changed. In case y_i is not chosen by the mechanism, the lie does not make any difference; in case y_i is selected, the cost of i can only be worse off. Hence, no one has incentive to lie and the mechanism is SP.

Let O be an optimal location and OPT is the optimal cost. The social cost induced by the mechanism is:

$$\begin{aligned}
SC &= \sum_{k=1}^N \frac{w_k}{W} \sum_{i=1}^N \sum_{j=1}^{w_i} d(y_k, x_{ij}) \leq \sum_{k=1}^N \frac{w_k}{W} \sum_{i=1}^N \sum_{j=1}^{w_i} \left(d(y_k, O) + d(O, x_{ij}) \right) \\
&\leq \sum_{k=1}^N w_k \cdot d(y_k, O) + \sum_{k=1}^N \frac{w_k}{W} \sum_{i=1}^N \sum_{j=1}^{w_i} d(O, x_{ij}) = \sum_{k=1}^N w_k \cdot d(y_k, O) + OPT \\
&\leq \sum_{k=1}^N \sum_{j=1}^{w_k} \left(d(y_k, x_{kj}) + d(x_{kj}, O) \right) + OPT \leq 3 \cdot OPT
\end{aligned}$$

where the last inequality is due to $\sum_{k=1}^N \sum_{j=1}^{w_k} d(y_k, x_{kj}) \leq \sum_{k=1}^N \sum_{j=1}^{w_k} d(O, x_{kj}) = OPT$. \square

Lemma 2. *Any randomized SP has approximation ratio at least $2 - 4N^{-1/3}$ even if each agent possesses only one location.*

Proof. Let n be a large number. Let $\epsilon = o(1)$ be a function of n and $0 < \gamma < 1$ be a constant, both are to be defined later. In the game there are $N = 2n$ agents. Consider a graph G (in Figure 1) consisting of $2n+2$ vertices $U \cup V \cup \{s, t\}$ where $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$. The set of allowable locations $\mathcal{F} = \{s, t\}$. The distances are given as follows.

$$\begin{aligned}
d(s, u_i) &= d(t, v_i) = 1 \quad \forall 1 \leq i \leq n, \\
d(u_i, v_i) &= 2\epsilon \quad \forall 1 \leq i \leq n.
\end{aligned}$$

Let f be a randomized SP mechanism. Consider a location profile \mathbf{x}^0 where there exists one agent on each vertex in $U \cup V$. For a vertex $w \in \mathcal{F}$, $\sum_{i=1}^n (d(w, u_i) + d(w, v_i)) = (2 + 2\epsilon)n$. Hence, there exists an agent whose expected cost is at least $1 + \epsilon$. Without loss of generality suppose that the agent locates in U .

Consider the following location profiles \mathbf{x}^k for $0 \leq k \leq n - n^\gamma$ defined recursively. In profile \mathbf{x}^k , there are k agents in s , $(n - k)$ agents in U and n agents stay in the same locations in V as the initial profile \mathbf{x}^0 . We argue that in profile \mathbf{x}^k , there exists an agent whose expected cost is at least $1 + \epsilon - \frac{1+\epsilon}{n-k}$. Without loss of generality, suppose that the agent locates in a vertex u_{k+1} . Profile \mathbf{x}^{k+1} is defined as the same as \mathbf{x}^k except that the agent previous in u_{k+1} now locates on s . Then, we prove that in profile \mathbf{x}^k for appropriate k , the approximation ratio is close to 2.

In \mathbf{x}^k , denote $p_s^k = \mathbb{P}[f(\mathbf{x}^k) = s]$, $p_t^k = \mathbb{P}[f(\mathbf{x}^k) = t]$ and $q^k = \mathbb{P}[f(\mathbf{x}^k) \in U \cup V]$. We show the following main lemma.

Lemma 3. *In profile \mathbf{x}^k , for $0 \leq k \leq n - n^\gamma$, there exists an agent located in U whose expected cost is at least $1 + \epsilon - (1 + \epsilon)(\mathcal{H}_n - \mathcal{H}_{n-k-1})$ where \mathcal{H}_i is the i^{th} Harmonic number. Moreover, in profile \mathbf{x}^k , for $1 \leq k \leq n - n^\gamma$, $p_s^k \leq p_t^k + \frac{1+\epsilon}{\epsilon}(\mathcal{H}_n - \mathcal{H}_{n-k})$.*

Proof. We prove the lemma by induction. For $k = 0$, we have proved that there exists an agent with cost at least $1 + \epsilon$. Assume that the hypothesis hold for location profile \mathbf{x}^{k-1} . In this profile, there exists an agent with expected cost at least $1 + \epsilon - (1 + \epsilon)(\mathcal{H}_n - \mathcal{H}_{n-k})$. W.l.o.g suppose that the agent locates in vertex u_k in profile \mathbf{x}^{k-1} . In profile \mathbf{x}^k , the agent is now locates in s . By strategy-proofness, $\mathbb{E}[d(u_k, f(\mathbf{x}^k))] \geq \mathbb{E}[d(u_k, f(\mathbf{x}^{k-1}))]$ since otherwise in profile \mathbf{x}^{k-1} , the agent may report its location as s and reduce the cost.

Suppose that $p_s^k > p_t^k + \frac{1+\epsilon}{\epsilon}(\mathcal{H}_n - \mathcal{H}_{n-k})$. Note that $p_s^k + p_t^k + q^k = 1$. We have

$$\begin{aligned} \mathbb{E}[d(u_k, f(\mathbf{x}^k))] &= p_s^k \cdot 1 + p_t^k \cdot (1 + 2\epsilon) + q^k \cdot (1 + \epsilon) = (1 + \epsilon) + \epsilon(p_t^k - p_s^k) \\ &< 1 + \epsilon - (1 + \epsilon)(\mathcal{H}_n - \mathcal{H}_{n-k}) \leq \mathbb{E}[d(u_k, f(\mathbf{x}^{k-1}))] \end{aligned}$$

contradiction to the strategy-proofness. Therefore, $p_s^k \leq p_t^k + \frac{1+\epsilon}{\epsilon}(\mathcal{H}_n - \mathcal{H}_{n-k})$.

In profile \mathbf{x}^k , consider the total cost of $(n - k)$ agents in U . This total cost is:

$$\begin{aligned} &p_s^k \cdot 1 \cdot (n - k) + p_t^k \cdot (1 + 2\epsilon) \cdot (n - k) + q^k \cdot (1 + \epsilon) \cdot (n - k - 1) = \\ &= (n - k)[1 + \epsilon + \epsilon(p_t^k - p_s^k) - q^k \cdot \frac{1 + \epsilon}{n - k}] \geq \\ &\geq (n - k)[1 + \epsilon - (1 + \epsilon)(\mathcal{H}_n - \mathcal{H}_{n-k-1})] \end{aligned}$$

where, again, we use $p^k + q^k = 1$ and inequality $p_s^k \leq p_t^k + \frac{1+\epsilon}{\epsilon}(\mathcal{H}_n - \mathcal{H}_{n-k})$. Thus, among all $(n - k)$ agents located in U , there is one whose (expected) cost is at least $1 + \epsilon - (1 + \epsilon)(\mathcal{H}_n - \mathcal{H}_{n-k-1})$. \square

Choose $k = \gamma n$. The social cost in profile \mathbf{x}^k is:

$$\begin{aligned} C(f, \mathbf{x}^k) &\geq p_s^k \cdot [(1 - \gamma)n + (1 + 2\epsilon)n] + p_t^k \cdot [(2 + 2\epsilon)\gamma n + (1 + 2\epsilon)(1 - \gamma)n + n] \\ &\quad + q^k[\gamma n + (1 + \epsilon)(2 - \gamma)n] \\ &> (2 + 2\epsilon)n + (p_t^k - p_s^k)\gamma n - \epsilon\gamma q^k \geq (2 + \epsilon)n + \gamma n \frac{1 + \epsilon}{\epsilon} \log(1 - \gamma) - \epsilon\gamma q^k \end{aligned}$$

where the second inequality is due to Lemma 3: $p^k \leq q^k + \frac{1+\epsilon}{\epsilon}(\mathcal{H}_n - \mathcal{H}_{n-k}) \leq q^k + \frac{1+\epsilon}{\epsilon} \log n^{1-\gamma}$ for n large enough. Besides, opening a facility at s has social cost at most $(1 + 2\epsilon)n + n^\gamma$. Hence, the approximation ratio is at least:

$$\frac{[2 + \epsilon - (1 + \epsilon)\epsilon]n}{(1 + 2\epsilon)n + n^\gamma} > 2 - 2\epsilon - n^{\gamma-1} - \frac{1 + \epsilon}{\epsilon n^\gamma} > 2 - \frac{4}{n^{1/3}}$$

where the last inequality is due to the choice of the parameters. Therefore, the approximation ratio is at least $2 - \frac{4}{N^{1/3}}$. \square

4 Randomized group strategy-proof mechanisms

In this section we study the performance of GSP mechanisms. First we give a deterministic $(2N + 1)$ -approximation mechanism which is inspired by the dictatorship mechanism in [12]. Although the mechanism is simple, we argue that it is the best that one could hope by showing a tight lower bound for any randomized GSP mechanism.

Deterministic mechanism: Given a location profile $\mathbf{x} = (x_1, \dots, x_N)$ where $x_i = \{x_{i1}, \dots, x_{iw_i}\}$ for $1 \leq i \leq N$. Let $y_i = \text{med}(x_i)$. Let $i^* := \arg \max_{1 \leq j \leq N} w_j$. Open the facility at y_{i^*} .

Theorem 2. *The deterministic mechanism is GSP that yields $(2N + 1)$ -approximation. Moreover, no randomized GSP mechanism has approximation ratio better than $N^{1-3\epsilon}$ for arbitrarily small $\epsilon > 0$ even if each agent possesses one location and the facility could be opened everywhere in the metric space.*

The theorem follows Lemma 4 and Lemma 7.

Lemma 4. *The mechanism is GSP that yields $(2N + 1)$ -approximation.*

Proof. It is obvious that the mechanism is GSP. Let O be an optimal location and OPT be the optimal social cost. Without loss of generality, assume that the facility is opened at y_1 . We bound the social cost as follows.

$$\begin{aligned} C(f, \mathbf{x}) &= \sum_{i=1}^N \sum_{j=1}^{w_i} d(y_1, x_{i,j}) \leq \sum_{i=1}^N \sum_{j=1}^{w_i} (d(y_1, O) + d(O, x_{i,j})) \\ &\leq OPT + \sum_{i=1}^N w_i \cdot d(y_1, O) \leq OPT + \sum_{i=1}^N w_1 \cdot d(y_1, O) \\ &\leq OPT + \sum_{i=1}^N \sum_{j=1}^{w_1} (d(y_1, x_{1,j}) + d(x_{1,j}, O)) \leq (2N + 1)OPT \end{aligned}$$

where the last inequality is due to the facts $\sum_{j=1}^{w_1} d(y_1, x_{1,j}) \leq OPT$ and $\sum_{j=1}^{w_1} d(x_{1,j}, O) \leq OPT$. \square

In the remaining of section, we prove the tight bound for any randomized GSP mechanism. Our lower bound works even in a restricted variant in which each agent owns only one location and the set of allowable locations \mathcal{F} (where the facility could be opened) is the set of all vertices of a given network. Hence, until the end of the section, we consider and prove lower bound on this restricted variant.

Starting point The lower bound $\Omega(N)$ for deterministic GSP mechanisms is devised from the characterization of Schummer and Vohra [12]. As dictatorship is the only deterministic GSP mechanism for arbitrary metric space, the lower bound is straightforwardly deduced. However, there is no similar characterization for randomized GSP mechanisms. In our approach, we start looking for a game which induces the same lower bound for deterministic GSP mechanisms without using the characterization in [12]. Consider the following instance. A network graph $G(U \cup V, E)$ consists of $2N$ vertices, where $U = \{u_1, \dots, u_N\}$ and $V = \{v_1, \dots, v_N\}$. Vertices in U form a complete graph with edge of distance 1. Each vertex v_i connects to all vertices in $U \setminus \{u_i\}$ by edge of cost $1 - \epsilon$. Consider an initial location profile \mathbf{x}^0 in which there are N agents, agent i locates on vertex u_i for $1 \leq i \leq N$. We study two cases.

Suppose that in profile \mathbf{x}^0 the facility is opened in U , w.l.o.g in u_1 . Consider location profile \mathbf{x}^1 in which agents $2, \dots, N$ locate vertex v_1 and agent 1 locates at u_1 . In this profile, the facility is not opened at v_1 since otherwise, agents $2, \dots, N$ have incentive to collaborate and move to vertex v_1 in the initial profile (they decrease their cost from 1 to $(1 - \epsilon)$). Hence, the social cost is at least $(N - 1)(1 - \epsilon)$ while the OPT is $2 - \epsilon$ by opening the facility at v_1 .

Suppose that in profile \mathbf{x}^0 the facility is open in V , w.l.o.g in v_1 . The cost of agent 1 is $2 - \epsilon$ and the cost of the other is $1 - \epsilon$. Consider location profile \mathbf{x}^2 in which agents $1, \dots, N - 1$ report vertex v_N and agent N reports u_N . In this profile, the facility is not opened at v_N since otherwise in profile \mathbf{x}^0 agents $1, \dots, N - 1$ have incentive to collaborate and move to vertex v_N (agent 1 decreases strictly her cost while the cost of the other in the cooperation remains unchanged). Again, the approximation ratio is larger than $(N - 1)/2$ in this profile.

Idea for lower bound of randomized mechanisms. The previous argument does not carry for randomized mechanisms. Consider the second case of the analysis in the previous paragraph. In profile \mathbf{x}^2 , a randomized mechanism may open a facility with probability arbitrarily close to 1 at vertex v_N and with the remaining probability (small but positive), open a facility in other vertex, for example v_2 , in order to increase the cost of one agent in the cooperation and so prevent the agent to participate from collaborating.

An idea to circumvent is the following. We modify the edge costs between U and V to break the symmetry. Then, argue that in some profiles with a bunch of agents in a vertex, any randomized GSP mechanism will open the facility at that vertex with large probability but there is still gap between this probability and 1. Using the gap, we amplify the approximation ratio.

Let $0 < \epsilon < 1$ be an arbitrarily small constant. Let n be a large integer and m be also an integer such that $(2m + 1)^m = n^{\epsilon/2}$, i.e $m = \Theta(\epsilon \log n / \log \log n)$. We can choose n, m such that $\ell = n/m$ is integer.

Lemma 5. Consider a sequence $(\beta_i)_{i=1}^m$ defined as follows:

$$\begin{aligned} \beta_m &= n^{-3\epsilon/2}, \\ \beta_i &= m\beta_{i+1} + (m + 1)\beta_m \quad \forall 1 \leq i \leq m - 1. \end{aligned}$$

Then, sequence $(\beta_i)_{i=1}^m$ is a decreasing, $\beta_1 \leq n^{-\epsilon}$ and

$$\beta_{i+1} + 2\beta_m = \frac{m-1}{m}\beta_m + \frac{1}{m}\beta_i$$

Proof. Clearly, by definition of the sequence, $(\beta_i)_{i=1}^m$ is decreasing and the last formula is straightforward. We prove that $\beta_1 < n^{-\epsilon}$. Again, by definition, $\beta_i \leq (2m + 1)\beta_{i+1}$ for all i . Therefore, $\beta_1 \leq (2m + 1)^m \cdot \beta_m = (2m + 1)^m n^{-3\epsilon/2} = n^{-\epsilon}$ (by the choice of m). \square

Consider a graph $G(U \cup V, E)$ consisting of $(n + m)$ vertices $U = U_1 \cup \dots \cup U_m$ where $U_i = \{u_{i1}, \dots, u_{i\ell}\}$, for $1 \leq i \leq m$, and $V = \{v_1, \dots, v_m\}$. Vertices in U form an independent set. Each vertex v_j is connected with all vertices in U such that the distances from v_j to any vertex in U_i are the same. Denote this distance as $d(v_j, U_i)$ (i.e $d(v_j, U_i) = d(v_j, u_{i1}) = \dots = d(v_j, u_{i\ell})$). We define the distances between vertices in U and V as follow. For all $1 \leq i, j \leq m$,

$$d(v_j, U_i) = 1 + \beta_{t(i,j)}$$

where $t(i, j) = 1 + (i - j \bmod m)$ and $(\beta_i)_{i=1}^{m-1}$ is defined in Lemma 5. As the distance from any vertex in U_i to any one in U_j is the same, we also denote such distance as $d(U_i, U_j)$. Note that

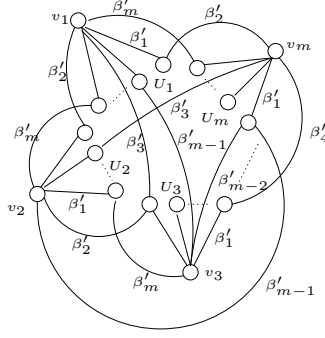


Figure 2: A part of graph G where $\beta'_i = 1 + \beta_i \forall 1 \leq i \leq m$.

$d(U_i, U_j)$ means the distance between two different vertices in U_i . By definition, the diameter of the graph is at most $2 + 2\beta_1$ and $2 + 2\beta_m \leq d(U_i, U_j) \leq 2 + 2\beta_1$.

Let f be a randomized GSP mechanism. Let \mathbf{x}^0 be a location profile in which there is one agent on each vertex in U . We prove the following main lemma.

Lemma 6. *There exists a location profile \mathbf{x} in which at least $(n - \ell - 1)$ agents locate on a vertex $v_i \in V$ for some i , and the others' locations are the same as in \mathbf{x}^0 such that $\mathbb{P}[f(\mathbf{x}) = v_i] < 1 - \beta_m$. Moreover, among all agents whose locations are the same as in \mathbf{x}^0 , at most three agents do not stay in U_i (in other words, almost such agents have locations in U_i).*

Proof. In profile \mathbf{x}^0 , let p_i be the probability that the facility is opened at agent i 's location for $1 \leq i \leq n$. Let q be the total probability that the facility is opened in U . We consider two cases where $q \geq 3\beta_1$ and $q < 3\beta_1$.

Case 1: $q \geq 3\beta_1$. Without loss of generality assume that $p_1 \leq \dots \leq p_n$. So $p_k \leq q/(n - k + 1)$. In profile \mathbf{x}^0 , the expected cost of agent k is at least $(1 - q) \cdot (1 + \beta_m) + q \cdot (1 - \frac{1}{n - k + 1}) \cdot (2 + 2\beta_m)$. Consider the profile \mathbf{x}^1 in which agents $1, \dots, n - 3$ locate at vertex v_i for some arbitrary i , and the others' locations are the same as in profile \mathbf{x}^0 . Let w_k be the location of agent k in profile \mathbf{x}^0 . By group-strategyproof, the mechanism f must guarantee the existence of $k \in [1, n - 3]$ such that $\mathbb{E}[d(f(\mathbf{x}^1), w_k)] > \mathbb{E}[d(f(\mathbf{x}^0), w_k)]$ since otherwise agents $1, 2, \dots, n - 3$ may collaborate, report together their locations as v_i and all get better off. Denote $\alpha_1 := \mathbb{P}[f(\mathbf{x}^1) = v_i]$. We have:

$$\mathbb{E}[d(f(\mathbf{x}^1), w_k)] \leq \alpha_1 \cdot (1 + \beta_1) + (1 - \alpha_1) \cdot (2 + 2\beta_1) \quad \forall 1 \leq k \leq n - 3$$

Hence, if there exists $1 \leq k \leq n - 3$ such that $\mathbb{E}[d(f(\mathbf{x}^1), w_k)] \geq \mathbb{E}[d(f(\mathbf{x}^0), w_k)]$ then:

$$\alpha_1 \cdot (1 + \beta_1) + (1 - \alpha_1) \cdot (2 + 2\beta_1) > (1 - q) \cdot (1 + \beta_m) + q \cdot (1 - \frac{1}{n - k + 1}) \cdot (2 + 2\beta_m)$$

As $k \leq n - 3$ and $q \geq 3\beta_1$, we deduce

$$\alpha_1 < 1 - \frac{q - \frac{2q}{(n - k + 1)} - \beta_1 + \beta_m}{1 + \beta_1} < 1 - \beta_m.$$

Case 2: $q < 3\beta_1$. Without loss of generality assume that in profile \mathbf{x}^0 , the probability that the facility is opened at v_1 is largest among all vertices in V . So $\mathbb{P}[f(\mathbf{x}^0) = v_1] \geq (1 - q)/m$. Let $a_1, \dots, a_{n - \ell}$ be agents in $U_1 \cup \dots \cup U_{m - 1}$ such that $p_{a_1} \leq p_{a_2} \leq \dots \leq p_{a_{n - \ell}}$. Remark that

$p_{a_k} \leq q/(n - \ell - k + 1)$. First, we bound the cost of agent a_k in profile \mathbf{x}^0 . Let z_k be the location of agent a_k and let $1 \leq i(k) \leq m - 1$ be an index such that $z_k \in U_{i(k)}$. The cost of agent a_k is:

$$\begin{aligned} c_{a_k}(f, \mathbf{x}^0) &= \mathbb{P}[f(\mathbf{x}^0) = v_1] \cdot d(U_{i(k)}, v_1) + \sum_{j \neq 1} \mathbb{P}[f(\mathbf{x}^0) = v_j] \cdot d(U_{i(k)}, v_j) + \\ &\quad + \sum_j \mathbb{P}[f(\mathbf{x}^0) = v_j] \cdot d(U_{i(k)}, U_j) \\ &> \frac{1-q}{m} \cdot (1 + \beta_{i(k)}) + \frac{(m-1)(1-q)}{m} \cdot (1 + \beta_m) + \left(q - \frac{q}{n - \ell - k + 1} \right) \cdot 2 \\ &> \left(1 + \frac{m-1}{m} \beta_m + \frac{1}{m} \beta_{i(k)} \right) (1 - q) + 2q - \frac{2q}{n - \ell - k + 1} \end{aligned}$$

Consider location profile \mathbf{x}^2 in which agents $a_1, \dots, a_{n-\ell-1}$ locate at v_m and the other agents' locations are the same as in profile \mathbf{x}^0 . Denote $\alpha_2 := \mathbb{P}[f(\mathbf{x}^2) = v_m]$. By group-strategyproof, there exists k such that $\mathbb{E}[d(f(\mathbf{x}^2), z_k)] > \mathbb{E}[d(f(\mathbf{x}^0), z_k)]$ since otherwise in profile \mathbf{x}^0 , agents $a_1, \dots, a_{n-\ell-1}$ have incentive to move together to v_m and all get better off. Note that distance $d(U_{i(k)}, v_m) = 1 + \beta_{i(k)+1}$. Then in profile \mathbf{x}^2 , we have $\mathbb{E}[d(f(\mathbf{x}^2), z_k)] < \alpha_2(1 + \beta_{i(k)+1}) + (1 - \alpha_2)(2 + 2\beta_1)$. As $\mathbb{E}[d(f(\mathbf{x}^2), z_k)] > \mathbb{E}[d(f(\mathbf{x}^0), z_k)]$ holds for some $k \in [1, n - \ell - 1]$, we have:

$$\alpha_2 (1 + \beta_{i(k)+1}) + (1 - \alpha_2) (2 + 2\beta_1) > \left(1 + \frac{m-1}{m} \beta_m + \frac{1}{m} \beta_{i(k)} \right) (1 - q) + 2q - \frac{2q}{2}$$

Therefore,

$$\alpha_2 < 1 - \frac{\left(\frac{m-1}{m} \beta_m + \frac{1}{m} \beta_{i(k)} \right) (1 - q) - \beta_{i(k)+1}}{1 + 2\beta_1 - \beta_{i(k)+1}} < 1 - \beta_m$$

where the last inequality is due Lemma 5, the case assumption $q < 2\beta_1$ and $\beta_{i(k)} \cdot \beta_1$ is dominated by β_m for any $i(k)$. □

Lemma 7. *There exists an instance with N agents in which any randomized GSP has approximation ratio at least $N^{1-3\epsilon}$ for $\epsilon > 0$ arbitrarily small constant.*

Proof. First, we construct recursively a family of graphs $H^j(\gamma_1, \dots, \gamma_m)$ for $j \geq 0$ where vertices are $(U^j \cup V^j)$ and $\gamma_1, \dots, \gamma_m$ are variables. In graph $H^j(\gamma_1, \dots, \gamma_m)$, the lengths of edges are taken from the set $\{1 + \gamma_1, \dots, 1 + \gamma_m\}$. Denote n_j be the number of vertices in graph H^j . Let n, m be large constant that are defined in the construction of graph G previously. Define graph $H^0(\gamma_1, \dots, \gamma_m)$ is the same as graph G described above where $U^0 = U$ and $V^0 = V$ except that now the lengths of edges are taken from the set $1 + \gamma_1, \dots, 1 + \gamma_m$. For example, if we assign variable $\gamma_i = \beta_i$ for $1 \leq i \leq m$ (where β_i is defined in Lemma 5) then $H^0(\gamma_1, \dots, \gamma_m) = H^0(\beta_1, \dots, \beta_m) = G$.

Intuitively, graph H^j contains m copies of graphs H^{j-1} and each of such copies plays similar role as vertices $U_i \cup v_i$ in the description of G . Formally, graph $H^j(\gamma_1, \dots, \gamma_m)$ consists of $n_j = mn_{j-1}$ vertices that we can partition the vertices as $U^j = U_1^j \cup \dots \cup U_m^j$ and $V^j = V_1^j \cup \dots \cup V_m^j$. For each $1 \leq i \leq m$, the restricted graph of $H^j(\gamma_1, \dots, \gamma_m)$ on $U_i^j \cup V_i^j$ is the same graph as $H^{j-1}(\gamma_1, \dots, \gamma_m)$. Moreover, each vertex in V_i^j connects with a vertex in $U_{i'}^j$ by an edge of length $1 + \gamma_{1+t(i,i')}$ (where $t(i, i') = 1 + (i - i' \bmod m)$) for $1 \leq i \neq i' \leq m$. Note that in graph $H^j(\gamma_1, \dots, \gamma_m)$, all edges have length in $\{1 + \gamma_1, \dots, 1 + \gamma_m\}$ and the diameter of the graph is at most $2 + 2\gamma_1$. Additionally, an invariant in any graphs is $|U^j| > |V^j|$.

Let t be a large constant to be defined later. Let $\beta_1^t, \dots, \beta_m^t$ be a sequence defined in Lemma 5 where parameter n in the lemma is replaced by $n_t/2$. Consider graph $H^t(\beta_1^t, \dots, \beta_m^t)$ and initial

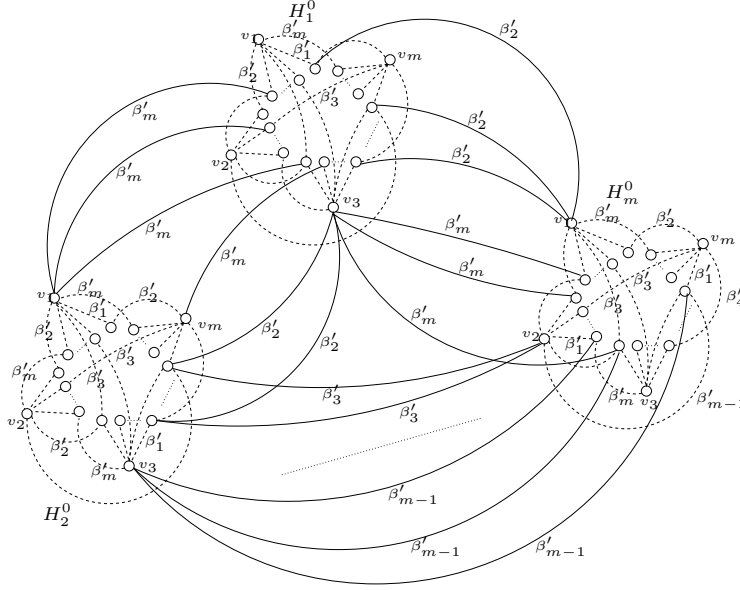


Figure 3: A part of graph $H^1(\beta_1^1, \dots, \beta_m^1)$ where in the picture $\beta'_i = 1 + \beta_i^1 \forall 1 \leq i \leq m$.

location profile \mathbf{x} in which there is one agent on each vertex in U^t . Let N be the number of agents ($N = |U^t|$). We have $N > n_t/2$ as $|U^t| > |V^t|$. Therefore, $\beta_m^t > N^{-2\epsilon}$. By Lemma 6, there exists a profile \mathbf{x}^1 in which at least $(N - N/m - 1)$ agents locate on a vertex $v \in V_{i_1}^t$ for some i_1 and the others' locations are the same as in \mathbf{x}^0 such that $\mathbb{P}[f(\mathbf{x}^1) = v] < 1 - \beta_m^t < 1 - N^{-2\epsilon}$. By the symmetry of the graph G^t , the statement is valid for any vertex $v \in V_{i_1}^t$. We denote A_1 the set of agents whose locations in \mathbf{x} and \mathbf{x}^1 are different.

Now consider graph H^t and profile \mathbf{x} restricted on vertices $U_{i_1}^t \cup V_{i_1}^t$. By construction, this is a graph H^{t-1} and we denote the profile restricted on this graph is \mathbf{x}' . Apply again Lemma 6 on the graph H^{t-1} and profile \mathbf{x}' , there exists there exists a profile \mathbf{x}^2 in which at least $(N/m - N/m^2 - 1)$ agents locate on a vertex $v \in V_{i_2}^{t-1}$ for some i_2 and the others' locations are the same as in \mathbf{x}' such that $\mathbb{P}[f(\mathbf{x}^2) = v] < 1 - \beta_m^t < 1 - N^{-2\epsilon}$. By the symmetry of the graph G^t , the statement is valid for any vertex $v \in V_{i_2}^t$. Remark that $V_{i_1}^t \supset V_{i_2}^{t-1}$. We denote A_2 the set of agents whose locations in \mathbf{x}' and \mathbf{x}^2 are different.

We apply the same argument by considering graph H^t and profile \mathbf{x} restricted on vertices $U_{i_2}^t \cup V_{i_2}^t$ and so on. In the last round, we get $V_{i_{t+1}}^0$.

Let $v^* \in V_{i_1}^t \cap V_{i_2}^{t-1} \cap \dots \cap V_{i_{t+1}}^0$. Consider graph G^t and a location profile \mathbf{x}^* in which agents in $(A_1 \cup \dots \cup A_t)$ locate at v^* and the others have the same locations as in \mathbf{x} . We have $\mathbb{P}[f(\mathbf{x}^*) = v^*] < 1 - \beta_m^t < 1 - N^{-2\epsilon}$ since otherwise, in \mathbf{x} , all agents in $(A_1 \cup \dots \cup A_t)$ will get better off by reporting together their location as v^* . Hence, the social cost is at least $N^{-2\epsilon}(N - N/m^{t+1} - t)$. The optimal solution opens facility at v^* with cost at most $2N/m^t + 3t$ where $3t$ comes from the fact that at each round r for $1 \leq r \leq t$, in the considered profile of round r , there are at most three agents neither in A_r nor located in $V_{i_{t-r}}^{t-r-1}$ (Lemma 6). Choose t large enough, say $t = (1 - \epsilon) \log_m N - 1$, the approximation ratio is at least $N^{1-3\epsilon}$. \square

5 Conclusion and Further Directions

In the paper, we have characterized the performance of randomized SP and GSP mechanisms in placing a single facility on graphs. An obvious open question is to close the gap of the approximation ratio for randomized SP mechanisms. (The upper bound is 3 and the lower

bound is approximately 2.)

An interesting direction is to study the game of opening k facilities. Various works [10, 5, 6] have considered the game where $k = 2$. Nevertheless, the performance of (G)SP mechanisms is far from clear. No deterministic SP mechanism with bounded approximation ratio is known for the game on general metric, even for $k = 2$. Lu et al. [6] have designed a randomized SP mechanism with constant approximation ratio for this special case. However, their mechanism seems not be carried over for $k \geq 3$. It is worth to note that a generalization of the network in Lemma 2 gives a lower bound 3 for any randomized SP mechanism in placing k facilities. The best known algorithm for the optimization problem k -median (without the desire of (G)SP property) have approximation ratio $3 + \epsilon$ [3] where ϵ is arbitrarily small. That shows a separation between the optimization problem and the one in the context of mechanism design without payment.

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