Improved Online Scheduling in Maximizing Throughput of Equal Length Jobs

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Abstract. Motivated by issues raised from data broadcast and networks using ATM and TCP/IP, we consider an online scheduling problem on a single machine. In the problem, each job i is revealed at release time r_i, has processing time p_i, deadline d_i and weight w_i. Preemption is allowed and there are two models of preemption: preemption with restart and preemption with resume. The goal is to maximize the *throughput* — the total weight of all jobs completed on time. In the paper, we consider the problem where all processing time of jobs are equal and present improved algorithms which achieve 4.24-competitive in both models of preemption.

1 Introduction

Data broadcast involves information distribution from a server to clients. The advantage of broadcasting technologies is that different users having the same request can be simultaneously satisfied by a broadcast. Information lies in a large range, from movies, soccer matches to stock market news, etc. Clients are also diverse and have different interests in certain moments. Hence, to maximize a given *quality of service* it is allowed that the server interrupts the currently broadcast page and starts a new one. Nevertheless, to satisfy a previously-interrupted page, the server has to broadcast again from the beginning.

ATM network has been designed to send telephone, radio, television communication as well as usual network data. In networks using ATM and TCP/IP, IP packets have to be split into small ATM cells and fed into the ATM networks. In general, packet sizes are bounded by the capacity of Ethernet, i.e. 1500 bytes, and in many cases they have the same length which equals the maximal capacity. The network transmits cells separately. In maximizing a given *quality of service*, it is allowed to stop transmitting cells of some packets and start sending cells of other ones. However, in contrast to data broadcast, in order to complete packets that still have remaining cells unsent, the network only needs to transmit the remaining instead of starting from the beginning.

Problem definition These applications can be formulated as an online scheduling problem on a single machine where each job i arrives online at its release time r_i, has processing time p_i, deadline d_i and weight w_i. The job's parameters are unknown until its arrival. All these quantities except possibly w_i are integer. The objective is to maximize *throughput*, which is the total weight of jobs completed on time. Preemption of jobs is allowed. Motivated by the above applications, we consider two models of preemption in the problem:

(i) the *preemptive model with restart*, where a job can be interrupted, but when it is scheduled again, it must be scheduled from the beginning
(ii) the *preemptive model with resume*, where in contrast, when a job is scheduled again, the previously done work can be resumed.

The problem under these two models of preemption can be denoted as $1|\text{online}-r_i|\sum w_i(1-U_i)$ and $1|\text{online}-r_i;pmtn|\sum w_i(1-U_i)$, respectively according to notation in [3].

1.1 Related work

It is known that, in both models of preemption, the deterministic competitive ratio is unbounded if jobs' processing times are arbitrary [1]. Dürr et al. [6] presented an algorithm that gave a tight bound $\Theta(p/\log p)$ in both models where the processing time of all jobs are bounded by p.

In the model of preemption with restart, a 5-competitive algorithm is given in [9, 2]. Zheng et al. [12] provided an improved algorithm Bar that was 4.56-competitive for jobs with equal processing time, arbitrary weights. Chrobak et al. [5] gave a tight 1.5-competitive deterministic algorithm for equal processing time, unit weight jobs.

In the model of preemption with resume, the best known upper bound and lower bound competitive ratio for the case of equal length and arbitrary weight jobs are 5 [6] and 2.59 [2, 6], respectively. For an interesting special case where jobs have unit processing time (i.e. $p_i = 1\ \forall i$), the problem (related to buffer management for packet switches [8]) is widely studied, and the deterministic competitive ratio lies between $\phi(\approx 1.618)$ [4] and 1.83 [7]. Another direction of research is to consider resource augmentation, and in [10] a deterministic online algorithm was presented which has constant competitive ratio provided that the algorithm is allowed a constant speedup of its machine compared to the adversary.

1.2 Our contribution

In the paper, we study the problem with jobs of equal processing time, arbitrary weight. The variant of equal processing time jobs has been widely studied (see [11, chapter 14]). We give algorithms which are 4.24-competitive for both models of preemption. The algorithms are essentially the same and the analysis are based on charging schemes. Roughly speaking, the issue is to solve the dilemma of choice between a lower-weighted job with imminent deadline and a higher-weighted job with later deadline.

In Section 2, we recall some standard notions. Then we illustrate the ideas that inspire the improved algorithms. In Section 3, we consider the model of preemption with restart in which we present the algorithm together with its intuition and the analysis. Even though the improvement is small, the algorithm helps in designing an algorithm with the same competitive ratio in the model of preemption with resume. In Section 4, we describe a 4.24-competitive algorithm in the latter model. The structure of the proof is similar to the one in the former model. However, the charging scheme is more subtle since the adversary may schedule jobs in different pieces and it presents the main difficulty of the proof in the model of preemption with resume.

2 Preliminaries

An algorithm is r-*competitive* if for any job sequence released by an adversary, the algorithm gain is at least r-fraction of the optimal offline solution where the whole sequence of jobs is known in advance.

Let p be the jobs' processing time, i.e. $p_i = p \ \forall i$. Consider an algorithm. Let C_i be the completion time of job i by the algorithm. Let $q_i(t)$ be the remaining processing time of job i for the algorithm at time t. When there is no confusion, we simply write q_i. In preemption with restart, if a job i is interrupted at some time t then $q_i(t+1) = p$.

We say that an algorithm schedules a job at time t meaning that the algorithm executes unit of this job in interval $[t, t+1)$. A job i is *pending* for the algorithm at time t if it has not been completed before and $r_i \leq t$ and $t + q_i(t) \leq d_i$. A job i is *urgent* at time t if $d_i < t + q_i(t) + p$.

In the model of preemption with restart, let $S_i(t)$ be the latest moment before t that an algorithm starts job i. In the model of preemption with resume, $S_i(t)$ denotes the latest moment $\tau < t$ such that at time τ, the algorithm schedules job i but at the previous moment $(\tau - 1)$, the algorithm schedules other job or it is an idle-time. Again, when there is no confusion, we simply write S_i.

Without loss of generality, assume that the adversary starts a job if and only if the job will be completed and the adversary schedules job in Earliest Deadline First manner. In the analysis, we abbreviate the algorithm and the adversary by Alg and Adv, respectively.

Starting point of improved algorithms. Consider the following algorithm for the model of preemption with restart. If there is no currently scheduled job, schedule the pending one with highest weight. Otherwise, if there is a new job i arriving with weight at least twice that of the currently scheduled job then interrupt the latter and schedule i.

This algorithm is 5-competitive [9, 2]. Observe that the algorithm considers the jobs' deadlines only in verifying whether jobs are pending; it totally ignores the correlation of that important parameter among the jobs. Hence, a better algorithm should be more involved in the deadlines of jobs, not only in verifying the pending property. An idea of improvement is that if a new released job is urgent, even if its weight is not large, one may delay the execution of the currently scheduled job and schedule the new job. However, postponing the heavy currently scheduled job might result in a lost of throughput if some new heavier job will be released later. A treatment is presented in [12] in which they handled implicitly the jobs' deadlines by turning them into a function of weight. In the paper, we deal explicitly with jobs' deadlines and present new algorithms with improved bounds.

3 A 4.24-competitive Algorithm for Preemption with Restart.

The algorithm A Let $1 < \beta < 3/2$ be a constant to be defined later. Initially, set $Q := \emptyset$ and $\alpha := 0$. Throughout the execution of the algorithm, at any time, Q stores the last job interrupted according to condition [$A2$] below.

At time t, if there is either no currently scheduled job or a job completion, then schedule the heaviest pending job. Otherwise, let j be the currently scheduled job. If there is no new released job, then continue to schedule j. Otherwise, let i be a new released job with heaviest weight. Job j is interrupted if one of the following conditions holds.

$A1$. **if:** $w_i \geq 2w_j$ and $w_i \geq 2^{\alpha} w(Q)$ where $w(Q)$ is the weight of job in Q. (Note that if $Q = \emptyset$ then $w(Q) = 0$ so the second inequality requirement is always satisfied.)
do: Schedule job i. Set $\alpha := 0$ and $Q = \emptyset$.

$A2$. **if:** $\alpha = 0$, $\beta w_j \leq w_i \leq 2w_j$, i is urgent and j can be scheduled later, i.e. $d_j \geq t + 2p$.
do: Schedule job i^* which is the heaviest among all urgent jobs, i.e., $i^* = \arg\max\{w_\ell : d_\ell < t + 2p\}$. Set $\alpha = 1$ and $Q := \{j\}$.

$A3$. **if:** i is urgent, $w_i \geq 2w_j + w_{j'}$ where j' is the job previously interrupted by j and there is no job ℓ satisfying the following conditions:

$$S_j(t) + 2p \leq d_\ell < t + 2p \quad \text{and} \quad w_\ell \geq w_j.$$

do: Schedule job i.

At any interruption, if $\alpha \geq 1$ then $\alpha := \alpha + 1$. If a job is completed then set $\alpha = 0$ and $Q = \emptyset$. Conventionally, if an interruption satisfies both conditions [$A1$] and [$A3$] then we refer that interruption to [$A1$].

Informally, the counter α indicates whether there exists an interruption of type [$A2$] which is not followed by an $A1$-interruption or a job completion. If there exists, α also indicates the number of job interruptions from the last job q interrupted according to [$A2$] and the set Q stores job q.

Intuition of the algorithm

– The first condition [$A1$] lies in the same spirit as the 5-competitive algorithms: if there is a new released job with heavy enough weight then schedule that new job. In the condition, we need one more inequality ($w_i \geq 2^{\alpha} w_Q$) compared to the 5-competitive algorithm to ensure that w_i can compensate previous interruptions of different types.

– The second condition $[A2]$ represents the idea that it is possible to delay the currently scheduled job if a new urgent job arrives even with non-heavy weight. However, we do not want to interrupt many consecutive jobs due to this condition since that may result in a small throughput. It is the reason we introduce a control variable α and condition $[A2]$ depends on α. Precisely, the algorithm does not allow two $A2$-interruptions without an $A1$-interruption or a job completion in between.
– The purpose of the last condition $[A3]$ is to handle a situation in which a new urgent heavy job arrives after an $A2$-interruption, but not heavy enough to interrupt the currently scheduled job according to $[A1]$. If there exists a job ℓ with properties described in $[A3]$ then scheduling i is not profitable since we can schedule ℓ later and scheduling i means that ℓ will be dropped out. Otherwise, it is beneficial to interrupt j and schedule i.

First, we make some observations before presenting the analysis of the algorithm. The proof of Observation 1 can be found in the appendix.

Observation 1 *1. Before any $A2$-interruption is either a job completion or an $A1$-interruption.*

2. Consider a sequence of consecutive $A3$-interruptions. Then, before the first ($A3$-) interruption of the sequence is an $A2$-interruption. Moreover, there are at most two $A3$-interruptions in the sequence. Consequently, α is always at most 3.

Lemma 1. *Consider a sequence of jobs $m, \ldots, 1, 0$ where job ℓ interrupts job $(\ell+1)$ for $1 \leq \ell \leq m-1$ and job 0 interrupts job 1 according to condition $[A1]$. Then*

1. $w_\ell \leq w_0 \cdot 2^{-\ell}$ for $0 \leq \ell \leq m$.
2. $w_{\ell'} \leq w_0 \cdot 2^{-\ell}$ where ℓ' be a job started by Adv *in $[S_{\ell+1}, S_\ell)$ and has not been completed by* Alg*. Consequently, the total weight of jobs which have not been completed by* Alg *and are started by* Adv *in $[S_m, S_0)$ is bounded by $2w_0 - 2w_m$.*

Proof. 1. We prove the claim by induction. It is straightforward for $\ell = 0$. Suppose that $w_\ell \leq w_0 \cdot 2^{-\ell}$. If job ℓ interrupts job $(\ell+1)$ according to $[A1]$ or $[A3]$ then $w_{\ell+1} \leq w_\ell \cdot 2^{-1} \leq w_0 \cdot 2^{-\ell+1}$. The remaining case is that ℓ interrupts $(\ell+1)$ according to $[A2]$. Consider the first $A1$-interruption after time S_ℓ. Assume that it is the interruption between jobs h and $(h+1)$. This interruption exists since $h = 0$ is a candidate. By definition of $[A1]$, $w_\ell \leq w_h \cdot 2^{h-\ell}$ (since $\alpha = \ell - h$). Moreover, by induction hypothesis, $w_h \leq w_0 \cdot 2^{-h}$. Therefore, $w_\ell \leq w_0 \cdot 2^{-\ell}$.

2. By contradiction, let ℓ be the largest index such that there exists a job ℓ' started by Adv in $[S_{\ell+1}, S_\ell)$, $w_{\ell'} > w_0 \cdot 2^{-\ell}$ and ℓ' is not completed before by Alg. Hence, for any job $h > \ell$, $w_h \leq w_0 \cdot 2^{-h} < w_{\ell'} \cdot 2^{\ell-h}$. As ℓ' has not been completed by Alg, ℓ' may interrupt job $(\ell+1)$ according to $[A1]$. Hence, job ℓ' is job ℓ. This contradicts that $w_\ell \leq w_0 \cdot 2^{-\ell} < w_{\ell'}$.

The total weighs of jobs which have not been completed by ALG and are started by ADV in $[S_m, S_0)$ is bounded by $\sum_{\ell=0}^{m-1} w_0 \cdot 2^{-\ell} \leq 2w_0 - 2^{-m}w_0 \leq 2w_0 - 2w_m$.

□

Analysis We analyze the competitiveness of the algorithm by a charging scheme. For convenience we renumber the jobs completed by the algorithm from 1 to n, such that the completion times are ordered $C_1 < \ldots < C_n$. Also we denote $C_0 = 0$. We divide the schedule of the algorithm into *phases* $[C_{i-1}, C_i)$, for $i = 1, \ldots, n$. We say that $[C_{i-1}, C_i)$ is the *phase of job* i for $i = 1, \ldots, n$. Consider the phase of job i. Let $f(i)$ be the first job started by ALG in this phase. Remark that, $f(i)$ is not necessarily completed by ALG.

The Charging Scheme

1. If a job i is scheduled by ADV and i has been completed by ALG before then charge w_i to job i (self-charge).
2. If a job j is started by ADV in the phase of job i and j has not been completed before by ALG, then charge w_j to i.
3. For each phase of job i, if there exists a next phase of job $i+1$ such that $S_{f(i+1)} = C_i$, i.e. meaning no idle-time between jobs i and $f(i+1)$, then charge $2w_{f(i+1)}$ from job i to job $i+1$.

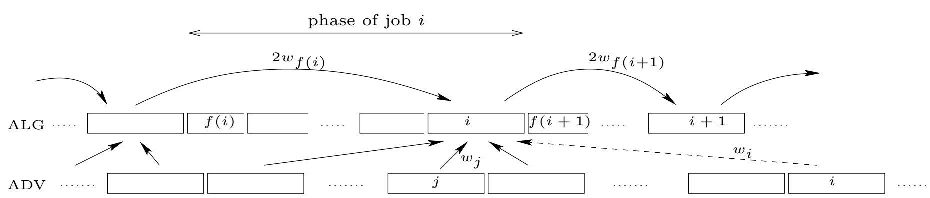

Fig. 1. Illustration of the charging scheme. The dashed pointer and the curly pointer represent the self-charge and the charge of step 3, respectively.

Informally, in the first two steps of the charging scheme, we charge the total throughput of ADV to jobs which are completed by ALG. In the third step, we redistribute the charges among the latter so that: each of such jobs receives a charge within factor r of its weight where r is the desired competitive ratio.

Consider the phase of job i and let i' be the job started by the ALG just after finishing i. Note that in case there exists no such job i', conventionally $w_{i'} = 0$; in case i' exists, $i' = f(i+1)$. In the analysis, we will argue that the total charge that i receives before step 3 of the charging scheme is at most $r \cdot w_i - 2w_{f(i)} + 2w_{i'}$ where r is a constant revealed later. In the redistribution step (step 3), each job i transfers $2w_{i'}$ to other job and possibly receives $2w_{f(i)}$. So the total charge that each job i completed by ALG receives is at most $r \cdot w_i$, which deduces the competitive ratio r of the algorithm. Hence, it is sufficient to prove the bound of job i's charge before step 3 by the term above. To simplify the exposition, until the end of the section, we refer the charge of a job as the amount that the job receives before redistribution (before step 3).

We say that a phase is of *type 1, 2, 3 or 0* if the last interruption in the phase is according to $[A1], [A2], [A3]$ or there is no interruption in the phase, respectively. If the phase of job i is 0-type then i receives at most one charge from a job j started in $[S_i, C_i)$ and probably one self-charge. As j does not interrupt i and j is not completed before, $w_j < 2w_i$. Hence, the total charge that i receives is at most $3w_i$. In the following, we bound the charge when the job i's phase is of type $1, 2$ and 3.

Lemma 2. *If the phase of job i is of type 1 then the charge that i receives is at most* $(3 + \beta)w_i - 2w_{f(i)} + 2w_{i'}$.

Proof. By Lemma 1, the charge that i receives from jobs started before S_i in Adv is at most $2w_i - 2w_{f(i)}$. Consider the job j scheduled by Adv in $[S_i, C_i)$. We have $w_j < 2w_i$ since otherwise, j can interrupt i by $[A1]$. If $w_j \leq \beta w_i$ or j has been completed by Alg or i receives no self-charge then the total charge that i receives is at most $\max\{2w_i - 2w_{f(i)} + w_i + \beta w_i, 2w_i - 2w_{f(i)} + w_i, 2w_i - 2w_{f(i)} + w_j\} \leq (3 + \beta)w_i - 2w_{f(i)}$. The remaining case is that $\beta w_i < w_j < 2w_i$ and i receives a self-charge. Let τ be the moment that Adv starts j.

1. **j is not urgent at τ**
   Then j is still pending at completion time of i, so $w_{i'} \geq w_j$. Hence, the charge that i receives in this case is $2w_i - 2w_{f(i)} + w_i + w_j \leq 3w_i - 2w_{f(i)} + w_{i'}$.
2. **j is urgent at τ**
   Since i receives a self-charge, $d_i \geq \tau + 2p$. As j is urgent and $w_j > \beta w_i$, i would have been interrupted by j according to condition $[A2]$, contradicts that the phase is of type 1. □

Lemma 3. *If the phase of job i is of type 2 then the charge that i receives is at most* $\max\left\{\frac{6}{\beta} - 1, 4\right\} \cdot w_i - 2w_{f(i)} + 2w_{i'}$.

Proof. Let i_1 be the job interrupted by i. By observation, before i_1 is either a job completion or an interruption of type $[A1]$. Then, by Lemma 1, the charge that i receives from jobs started by Adv before S_{i_1} is at most $2w_{i_1} - 2w_{f(i)}$. Due to the fact that all jobs have the same length, there are at most two jobs j_1 and j started by Adv in intervals $[S_{i_1}, S_i)$ and $[S_i, C_i)$, respectively (Figure 2). As i interrupts i_1 according to $[A2]$, at time C_i job i_1 is still pending. So at that moment, Alg will schedule some job i' with weight $w_{i'} \geq w_{i_1}$. Moreover, $d_i < S_i + 2p$ so either i receives no self-charge or $j = i$. Hence, the charge that i receives is at most $2w_{i_1} - 2w_{f(i)} + w_{j_1} + \max\{w_i, w_j\}$. In the following, we prove $2w_{i_1} + w_{j_1} + \max\{w_i, w_j\} \leq \max\left\{\frac{6}{\beta} - 1, 4\right\} \cdot w_i + 2w_{i'}$.

We have $w_{j_1} < 2w_{i_1}$ since otherwise, job j_1 would have interrupted job i_1 by condition $[A1]$. If $w_j \leq 2w_i$ then $2w_{i_1} + w_{j_1} + \max\{w_i, w_j\} \leq 2w_{i_1} + 2w_{i_1} + 2w_i \leq 4w_i + 2w_{i'}$. Consider the case $w_j > 2w_i$. Let $\tau \in [S_i, C_i)$ be the moment that Adv

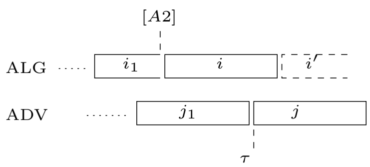

Fig. 2. Illustration of a phase of type 2

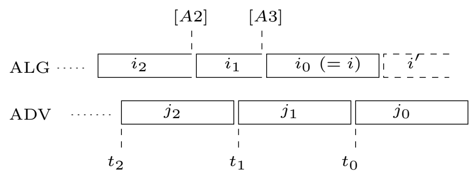

Fig. 3. Illustration of the phase with one interruption [$A3$]

starts j. If j is not urgent at τ then $w_{i'} \geq w_j$, so $2w_{i_1} + w_{j_1} + w_j < 4w_i + 2w_{i'}$. In the remaining, $w_j > 2w_i$ and j is urgent at time τ.

1. **There exists job ℓ such that $S_i + 2p \leq d_\ell < \tau + 2p$ and $w_\ell \geq w_i$.** In this case, we have $w_{i'} \geq w_\ell \geq w_i$. If $w_{j_1} \leq w_i$ then $w_{i'} \geq w_{j_1}$. If $w_{j_1} > w_i$ then j_1 is not urgent at its starting time by the ADV since otherwise, j_1 would have interrupted i_1 according to [$A2$]. Hence, j_1 is pending at S_i. By the job selection in condition [$A2$], j_1 is not urgent at S_i. Therefore, j_1 is also pending at time C_i, so $w_{i'} \geq w_{j_1}$. In both cases, $w_{i'} \geq w_{j_1}$. Besides, as j did not interrupt i according to [$A1$] $w_j \leq \max\{2w_i, 4w_{i_1}\}$. Thus, $w_j \leq 4w_{i_1}$. We have:

$$2w_{i_1} + w_{j_1} + w_j \leq 6w_{i_1} + w_{i'} \leq (6/\beta - 1) \cdot w_i + 2w_{i'}$$

2. **There does not exist job ℓ such that $S_i + 2p \leq d_\ell < \tau + 2p$ and $w_\ell \geq w_i$.** As j is urgent, $w_j < 2w_i + w_{i_1}$ since otherwise j would have interrupted job i by [$A3$]. Hence,

$$2w_{i_1} + w_{j_1} + w_j < 2w_{i_1} + w_{j_1} + 2w_i + w_{i_1}$$

If $d_{j_1} \geq S_i + 2p$ then $w_{i'} \geq \max\{w_{i_1}, w_{j_1}\}$, so $2w_{i_1} + w_{j_1} + 2w_i + w_{i_1} < 4w_i + 2w_{i'}$ (note that $w_i > w_{i_1}$). If $S_i + p < d_{j_1} < S_i + 2p$ and as j_1 is not completed before by ALG then by the choice of scheduled job at interruption [$A2$], $w_{j_1} \leq w_i$. If $d_{j_1} < S_i + p$ then we also have $w_{j_1} \leq w_i$ since otherwise, j_1 would have interrupted i_1 according to [$A2$]. Therefore, we also have $2w_{i_1} + w_{j_1} + 2w_i + w_{i_1} \leq 4w_i + 2w_{i'}$. □

Lemma 4. *If the phase of job i is of type 3 then the charge that i receives is at most* $\max\left\{\frac{3\beta+11}{2\beta+1}, 4 + \frac{3-\beta}{5\beta+2}\right\} w_i - 2w_{f(i)} + 2w_{i'}$.

Proof. By observation, there are at most two consecutive interruptions [$A3$] in the end of job i's phase. We consider first the case with one interruption. The case of two interruptions is analyzed similarly.

*1. **There is only one** $A3$**-interruption.*** Let $i_0(=i), i_1, i_2$ be jobs started by ALG in the phase such that i_0, i_1 interrupt i_1, i_2 according to [$A3$] and [$A2$], respectively. Let j_0, j_1 and j_2 be jobs started by ADV in $[S_{i_0}, C_{i_0})$, $[S_{i_1}, S_{i_0})$ and $[S_{i_2}, S_{i_1})$, respectively (Figure 3). By Lemma 1, the charge of jobs started by ADV from the beginning of the phase to S_{i_2} is at most $2w_{i_2} - 2w_{f(i_0)}$. As job i_0 is new released and urgent at time S_{i_0}, i_0 receives no self-charge or $j_0 = i_0$. In both cases, the total charge is bounded by $2w_{i_2} - 2w_{f(i_0)} + w_{j_2} + w_{j_1} + \max\{w_{j_0}, 2w_{i_0}\}$. In the following, we argue that $W := 2w_{i_2} + w_{j_2} + w_{j_1} + \max\{w_{j_0}, 2w_{i_0}\} \leq \max\left\{\frac{3\beta+11}{2\beta+1}, 4\right\} w_{i_0} + 2w_{i'}$.
Remark that j_0 does not interrupt i_0 according to [$A1$], so $\max\{w_{j_0}, 2w_{i_0}\} \leq \max\{8w_{i_2}, 4w_{i_1}, 2w_{i_0}\} \leq \max\{2, 8/(2\beta+1)\}w_{i_0} \leq 8/(2\beta+1)w_{i_0}$ (because $\beta < 3/2$). Note that $\max\{8w_{i_2}, 4w_{i_1}, 2w_{i_0}\} \leq 2w_{i_0} + 2w_{i_2}$, so sometimes, we only need a weaker inequality $\max\{w_{j_0}, 2w_{i_0}\} \leq 2w_{i_0} + 2w_{i_2}$. We also use inequalities $w_{j_2} < 2w_{i_2}$ (because j_2 does not interrupt i_2 according to [$A1$]), and $(2\beta+1)w_{i_2} \leq w_{i_0}$.

(a) **Case** $w_{j_1} > w_{i_0}$**.** By condition [$A3$], we have either $d_{j_1} \geq S_{i_0} + 2p$ or $d_{j_1} < S_{i_1} + 2p$ (otherwise, j_1 would have played the role of job ℓ at time $t = S_{i_0}$ in the definition of [$A3$]). In the former, $w_{i'} \geq w_{j_1}$. Hence,

$$4w_{i_0} + 2w_{i'} > 2w_{i_0} + 2(2w_{i_1} + w_{i_2}) + w_{j_1} > \\ > (2w_{i_0} + 2w_{i_2}) + w_{j_1} + 2w_{i_2} + 2w_{i_2} > W.$$

In the latter, job j_1 is urgent at its starting time t_1 by the ADV. Moreover, we know that no job ℓ satisfying $S_{i_1} + 2p \leq d_\ell < S_{i_0} + 2p$ and $w_\ell \geq w_{i_1}$ by definition of [$A3$]. So there is no job ℓ satisfying $S_{i_1} + 2p \leq d_\ell < t_1 + 2p$ and $w_\ell \geq w_{i_1}$. As $w_{j_1} > w_{i_0}$, j_1 would have interrupted i_1 according to condition [$A3$] (contradiction).

(b) **Case** $w_{j_1} \leq w_{i_0}$ **and** $w_{j_2} > w_{i_1}$**.** Then j_2 is not urgent at its starting time t_2 by the ADV (since otherwise, j_2 would have interrupted i_2). By job selection in [$A2$], j_2 is not urgent at S_{i_1} neither, i.e. $d_{j_2} \geq S_{i_1} + 2p$. Furthermore, by definition of condition [$A3$], we have $d_{j_2} \geq S_{i_0} + 2p$ since otherwise j_2 would played the role of job ℓ in the definition of [$A3$]. Hence, $w_{i'} \geq w_{j_2} > w_{i_1} > w_{i_2}$. Therefore,

$$4w_{i_0} + 2w_{i'} > w_{j_1} + 2w_{i_0} + (2w_{i_1} + w_{i_2}) + w_{j_2} + w_{i_2} > \\ > w_{j_1} + (2w_{i_0} + 2w_{i_2}) + w_{j_2} + 2w_{i_2} \geq W$$

(c) **Case** $w_{j_1} \leq w_{i_0}$ **and** $w_{j_2} \leq w_{i_1}$**.** We have:

$$W = w_{j_1} + w_{j_2} + w_{j_0} + 2w_{i_2} < w_{i_0} + w_{i_1} + \frac{8}{2\beta+1}w_{i_0} + 2w_{i_2} < \\ < w_{i_0} + (w_{i_1} + \frac{1}{2}w_{i_2}) + \frac{8}{2\beta+1}w_{i_0} + \frac{3}{2(2\beta+1)}w_{i_0} \leq \frac{3\beta+11}{2\beta+1} \cdot w_{i_0}$$

*2. **There are two** $A3$**-interruptions.*** (see Appendix). □

Theorem 1. *The algorithm is $(2+\sqrt{5})(\approx 4.24)$-competitive while $\beta = \sqrt{5} - 1$.*

Proof: Due to previous lemmas, the charge that a job i completed by ALG is at most $r = \max\{4 + \frac{3-\beta}{5\beta+2}, 3+\beta, \frac{6}{\beta} - 1, \frac{3\beta+11}{2\beta+1}\}$. The ratio r attains minimum value $(2+\sqrt{5})$ while $\beta = \sqrt{5} - 1$. □

4 A 4.24-competitive Algorithm for Preemption with Resume.

The algorithm is similar to algorithm A but the conditions depend on the remaining processing time of jobs rather than their entire processing time.

The algorithm B Initially, set $Q := \emptyset$ and $\alpha := 0$. At time t, if there is either no currently scheduled job or a job completion, then schedule the heaviest job. Otherwise, let j be the currently scheduled job. If there is no new released job, then continue to schedule j. Otherwise, let i be a new released job with heaviest weight. Job j is interrupted if one of the following conditions holds.

$B1$. **if:** $w_i \geq 2w_j$ and $w_i \geq 2^{\alpha} w(Q)$.
do: Schedule job i. Set $\alpha := 0$ and $Q = \emptyset$.

$B2$. **if:** $\alpha = 0$, $\beta w_j \leq w_i \leq 2w_j$, i is urgent and j can be scheduled later, i.e. $d_j \geq t + p + q_j$,
do: Schedule job i^* which is the heaviest among all urgent jobs, i.e., $i^* = \arg\max\{w_\ell : d_\ell \leq t + p + q_\ell\}$. Set $\alpha = 1$ and $Q := \{j\}$.

$B3$. **if:** i is urgent, $w_i \geq 2w_j + w_{j'}$ where j' is the job previously preempted by j and there is no another job ℓ satisfying the following conditions:

$$S_j + p + q_\ell(t) \leq d_\ell < t + p + q_\ell(t) \quad \text{and} \quad w_\ell \geq w_j.$$

do: Schedule job i.

At any interruption, if $\alpha \geq 1$ then $\alpha := \alpha + 1$. If a job is completed then set $\alpha = 0$ and $Q = \emptyset$. Conventionally, if an interruption satisfies both conditions [$B1$] and [$B3$] then we refer that interruption to [$B1$].

Analysis The charging scheme in the model of restart does not carry over since now the adversary may schedule jobs in different pieces. We present a more subtle charging scheme in order to analyze the algorithm by exploiting the equal-length property of jobs. Note that the algorithm considers the remaining processing time of jobs. Hence, at some points, the picture looks like the model in which jobs have arbitrary length. However, initially all jobs have the same length. This property is used in proving Lemma 5, the main lemma which is not valid if jobs have arbitrary length.

The Charging Scheme Again we renumber the jobs completed by the algorithm from 1 to n, such that the completion times are ordered $0 = C_0 < C_1 < \ldots < C_n$. For every $i = 1, \ldots, n$ we divide $[C_{i-1}, C_i)$ further into intervals: Let $a = \lceil (C_i - C_{i-1})/p \rceil$. The first interval is $[C_{i-1}, C_i - (a-1)p)$. The remaining intervals are

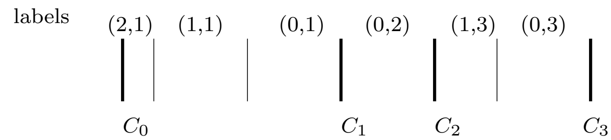

Fig. 4. The intervals as used by the charging procedure.

$[C_i - (b+1)p, C_i - bp)$ for every $b = a-2, \dots, 0$. We label every interval I with a pair (b,i) such that $I = [s, C_i - bp)$ for $s = \max\{C_{i-1}, C_i - (b+1)p\}$.

The charging also consists of three steps in which the first and the last steps are the same as the ones of the charging scheme in Section 3. The second step will be done by the following procedure, which maintains for every interval $[s,t)$ a set of jobs P that are started before t by the adversary and that are not yet charged to some job of the algorithm.

Initially $P = \emptyset$.
For all intervals $[s,t)$ as defined above in left to right order, **do**

- Let (b,i) be the label of the interval.
- Add to P all jobs j started by the adversary in $[s,t)$.
- If P is not empty, then remove from P the job j with the smallest deadline and charge it to i. Mark $[s,t)$ with j.
- If P is empty, then leave $[s,t)$ unmarked.
- Denote by P_t the current content of P.

Note that, no job i receives any charge from job started after C_i except self-charge. Using the fact that all jobs have the same length, we prove the following main lemma.

Lemma 5. *For every interval $[s,t)$, all jobs $j \in P_t$ are still pending for the algorithm at time t.*

Proof: Assume that P_t is not empty, and let j be the job in P_t with the smallest deadline. First we claim that there is a time s_0, such that every interval contained in $[s_0, t)$ is marked with some job j' satisfying $s_0 \le r_{j'}$ and $d_{j'} \le d_j$.

Let $[s', t')$ be the interval where the adversary started j. So j entered P by the charging procedure at that interval. Job j was in P during all the iterations until $[s,t)$, so every interval between s' and t is marked with some job of deadline at most d_j. Let $\mathcal{M}$ be the set of these jobs. If for every $j' \in \mathcal{M}$ we have $s' \le r_{j'}$, choose $s_0 = s'$ and we are done. Otherwise let $j' \in \mathcal{M}$ be the job with smallest release time. So $r_{j'} < s'$. Let $[s'', t'')$ be the interval where the adversary started j'. By the same argment as above, during the iteration over the intervals between s'' and s', job j' was in P. Therefore every such interval was marked with some job with deadline at most $d_{j'} \le d_j$. Now we repeat for s'' the argument we had for s'. Eventually we obtain a valid s_0, since P was initially empty. That proves the claim.

Now let $\mathcal{M}$ be the set of jobs charged during all intervals in $[s_0, t)$. So $j \notin \mathcal{M}$. In an Earliest Deadline First schedule of the adversary, job j would be completed not before $s_0 + (|\mathcal{M}| + 1)p$. But any interval has size at most p, so $t - s_0 \leq |\mathcal{M}|p$. We conclude that $d_j \geq t + p$, which shows that j is still pending for the algorithm at time t. □

Using Lemma 5, the remaining analysis follows the same structure as in the model of restart, though with different details. The lemmas and their proofs are presented in the appendix.

Theorem 2. *The algorithm B is* $(2+\sqrt{5})(\approx 4.24)$*-competitive when* $\beta = \sqrt{5}-1$.

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Appendix

A Model of Preemption with Restart

Observation 1 *1. Before any A2-interruption is either a job completion or an A1-interruption.*

2. Consider a sequence of consecutive A3-interruptions. Then, before the first (A3-) interruption of the sequence is an A2-interruption. Moreover, there are at most two A3-interruptions in the sequence. Consequently, α is always at most 3.

Proof. 1. By contradiction, suppose that i_1 is interrupted by i_2 and i_2 is interrupted according to condition $[A2]$ by i_3. Then the interruption of i_1 by i_2 is not of type $[A2]$ since otherwise counter $\alpha \geq 1$ so i_3 cannot interrupt i_2 according to condition $[A2]$. Moreover, if i_2 interrupts i_1 according to $[A3]$, meaning i_2 is urgent then i_3 cannot interrupt i_2 according to condition $[A2]$ because i_2 may not be scheduled later. Therefore, the interruption between i_1 and i_2 is of type $[A1]$.

2. Before the first interruption (of type $[A3]$) in the sequence must be an interruption of type $[A2]$ since otherwise that $A3$-interruption would have been referred to an interruption of type $[A1]$ as convention. We argue the second claim by contradiction. Suppose that there are n jobs $i_n, i_{n-1}, \ldots, i_0$ where $n \geq 3$ and i_ℓ interrupt $i_{\ell+1}$ according to $[A3]$ for $0 \leq \ell \leq n-1$. Hence, by previous claim, there exits a job i_{n+1} which is interrupted by i_n according to $[A2]$. We argue that in fact, i_0 is heavy enough to interrupt i_1 according to $[A1]$. As $n \geq 3$, we have:

$$\begin{aligned} w_{i_{n-3}} &\geq 2w_{i_{n-2}} + w_{i_{n-1}} \geq 2(2w_{i_{n-1}} + w_{i_n}) + w_{i_{n-1}} \\ &\geq 5(2w_{i_n} + w_{i_{n+1}}) + 2w_{i_n} > 2^4 w_{i_n} \end{aligned}$$

where the inequalities are due to condition of $[A3]$ and $w_{i_n} > w_{i_{n+1}}$. Hence, i_{n-3} satisfies condition $[A1]$ so the interruption between i_{n-3} and i_{n-2} would have been referred to $[A1]$ by convention (contradiction).
The claim that $\alpha \leq 3$ is straightforward from previous observations.

□

Here we present the remaining proof of Lemma 4.

Lemma 4. *If the phase of job i is of type 3 then the charge that i receives is at most* $\max\left\{\frac{3\beta+11}{2\beta+1}, 4 + \frac{3-\beta}{5\beta+2}\right\} w_i - 2w_{f(i)} + 2w_{i'}$.

Proof. **There are two interruptions of type $[A3]$.**

Let $i_0(=i), i_1, i_2, i_3$ be jobs started by Alg in the phase such that i_0, i_1, i_2 interrupt i_1, i_2, i_3 according to $[A3], [A3]$ and $[A2]$, respectively. Let j_0, j_1, j_2 and j_3 be jobs started by Adv in $[S_{i_0}, C_{i_0}), [S_{i_1}, S_{i_0}), [S_{i_2}, S_{i_1})$ and $[S_{i_3}, S_{i_2})$, respectively (Figure 5). By Lemma 1, the charge of jobs started by Adv from

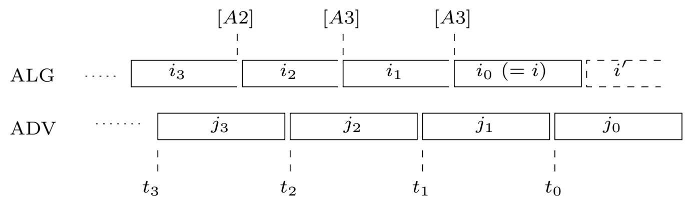

Fig. 5. Illustration for the phase with two interruptions [$A3$]

the beginning of the phase to S_{i_3} is at most $2w_{i_3} - 2w_{f(i)}$. As job i_0 is new released and urgent at time S_{i_0}, either i_0 receives no self-charge or if i_0 receives a self-charge then $w_{j_0} < w_{i_0}$. In the latter, the total charge of i_0 is at most $2w_{i_2} - 2w_{f(i)} + w_{j_3} + w_{j_2} + w_{j_1} + w_{j_0} + w_{i_0} < 2w_{i_2} - 2w_{f(i)} + w_{j_2} + w_{j_1} + 2w_{i_0}$. In the former, i_0 receives at most $2w_{i_2} - 2w_{f(i)} + w_{j_3} + w_{j_2} + w_{j_1} + w_{j_0}$. In the following, we argue that $2w_{i_3} + w_{j_3} + w_{j_2} + w_{j_1} + \max\{w_{j_0}, 2w_{i_0}\} \leq \max\left\{\frac{3\beta+11}{2\beta+1}, 4 + \frac{3-\beta}{5\beta+2}\right\} w_{i_0} + 2w_{i'}$.

Remark that j_0 does not interrupt i_0 according to [$A1$], so $\max\{w_{j_0}, 2w_{i_0}\} \leq \max\{2w_{i_0}, 4w_{i_1}, 8w_{i_2}, 16w_{i_3}\} \leq 2w_{i_0} + 2w_{i_2}$. Similarly, $w_{j_2} \leq \max\{2w_{i_2}, 4w_{i_3}\}$ and $w_{j_3} \leq 2w_{i_3}$.

(a) **Case $w_{j_1} > w_{i_0}$.** Then by condition [$A3$], we have either $d_{j_1} \geq S_{i_0} + 2p$ or $d_{j_1} < S_{i_1} + 2p$. In the former, $w_{i'} \geq w_{j_1}$. Hence,

$$\begin{aligned} 4w_{i_0} + 2w_{i'} &> 2w_{i_0} + w_{j_1} + 3(2w_{i_1} + w_{i_2}) \\ &> (2w_{i_0} + 2w_{i_2}) + w_{j_1} + (2w_{i_2} + 2w_{i_3}) + 2w_{i_3} + 2w_{i_3} \\ &> w_{j_0} + w_{j_1} + w_{j_2} + w_{j_3} + 2w_{i_3} \end{aligned}$$

In the latter, job j_1 is urgent at its starting time t_1 by the ADV. Moreover, we know that no job ℓ satisfying $S_{i_1} + 2p \leq d_\ell < S_{i_0} + 2p$ and $w_\ell \geq w_{i_1}$ by definition of [$A3$]. So there is no job ℓ satisfying $S_{i_1} + 2p \leq d_\ell < t_1 + 2p$ and $w_\ell \geq w_{i_1}$. As $w_{j_1} > w_{i_0}$, j_1 would have interrupted i_1 according to condition [$A3$] (contradiction).

(b) **Case $w_{j_1} \leq w_{i_0}$ and $w_{j_2} > w_{i_1}$.** Then j_2 is not urgent at its starting time t_2 by the ADV since otherwise, j_2 would have preempted i_2 according to [$A3$]. By [$A2$], $d_{j_2} \geq S_{i_1} + 2p$. Furthermore, by definition of condition [$A3$], we have $d_{j_2} \geq S_{i_0} + 2p$. Hence, $w_{i'} \geq w_{j_2} > w_{i_1}$. Therefore,

$$\begin{aligned} 4w_{i_0} + 2w_{i'} &> 2w_{i_0} + w_{j_1} + w_{i_0} + w_{j_2} + w_{i_1} \\ &> (2w_{i_0} + 2w_{i_2}) + w_{j_1} + w_{j_2} + w_{i_0} + w_{i_3} \\ &> w_{j_0} + w_{j_1} + w_{j_2} + w_{j_3} + 2w_{i_3} \end{aligned}$$

(c) **Case $w_{j_1} \leq w_{i_0}$ and $w_{j_2} \leq w_{i_1}$.** We have:

$$\begin{aligned} w_{j_0} + w_{j_1} + w_{j_2} + w_{j_3} + 2w_{i_3} &\leq (2w_{i_0} + 2w_{i_2}) + w_{i_0} + w_{i_1} + 4w_{i_3} \\ &\leq 3w_{i_0} + 2w_{i_1} + 3w_{i_3} \leq 4w_{i_0} + (3-\beta)w_{i_3} \\ &\leq \left(4 + \frac{3-\beta}{5\beta+2}\right) w_{i_0} \end{aligned}$$

where the last inequality is due to $w_{i_3} \leq \frac{1}{5\beta+2} w_{i_0}$ (by combining $w_{i_0} \geq 2w_{i_1} + w_{i_2}$, $w_{i_1} \geq 2w_{i_2} + w_{i_3}$ and $w_{i_2} \geq \beta w_{i_3}$).

□

B Models of Preemption with Resume

The following lemma is of the same spirit as Lemma 1.

Lemma 6. *Consider consecutive intervals $(m, i), (m-1, i), \ldots, (b+1, i), (b, i)$ in the phase of job i. Let j_ℓ be the last job scheduled by* ALG *in interval (ℓ, i) for $b \leq \ell \leq m$. If the last interruption in interval (b, i) is $[B1]$, i.e., job j_0 interrupts some job in (b, i) according to $[B1]$, then*

1. *$w_{j_\ell} \leq w_{j_b} \cdot 2^{-\ell+b}$ for $b \leq \ell \leq m$.*
2. *Let j'_ℓ be the job charged to i by the procedure while considering interval (ℓ, i) for $b \leq \ell \leq m$. We have $w_{j'_\ell} \leq w_{j_0} \cdot 2^{-\ell+b+1}$.*
   Consequently, the total charge that i receives from $(m, i), (m-1, i), \ldots, (b+1, i)$ is bounded by $2w_{j_0} - 2w_{j_m}$.

Proof: Using Lemma 5, the proof is similar to the one of Lemma 1. □

Let i' be the job started by the ALG just after finished i (at time C_i). Similarly to the analysis of algorithm A, in the following, we prove that before step 3 each job i receives a charge at most $rw_i - 2w_{f(i)} + 2w_{i'}$ where r is revealed later. Until the end of the section, we refer the charge of a job is the amount that the job receives before redistribution (before step 3 of the charging scheme).

We say that a phase is of *type 1, 2, 3 or 0* if the last interruption in the phase is according to $[B1], [B2], [B3]$ or there is no interruption in the phase, respectively . If the phase of job i is 0-type then i receives at most one charge from a job j, which marks the interval $(0, i)$, and probably one self-charge. As j does not interrupt i and j is not completed before, $w_j < 2w_i$. Hence, the total charge that i receives is at most $3w_i$. In the following, we bound the charge when the job i's phase is of type $1, 2$ and 3. Note that, for these cases, the interval $(0, i)$ is $[S_i, C_i)$ which has length p.

Lemma 7. *If the phase of job i is of type 1 then the charge that i receives is at most* $\max\{3+\beta, 4\} w_i - 2w_{f(i)} + 2w_{i'}$.

Proof. By Lemma 6, the charge from marked jobs of all intervals but $(0, i)$ to job i is at most $2w_i - 2w_{f(i)}$. Let j be a marked job of interval $(0, i)$. We have $w_j < 2w_i$ since otherwise, j can interrupt i by $[B1]$. If $w_j \leq \beta w_i$ or j has been completed by ALG or i receives no self-charge then the total charge that i receives, including the self-charge and the charge from j, is at most $\max\{2w_i - 2w_{f(i)} + w_i + w_j, 2w_i - 2w_{f(i)} + w_i, 2w_i - 2w_{f(i)} + w_j\} \leq (3+\beta)w_i - 2w_{f(i)}$. The remaining case is that $\beta w_i < w_j < 2w_i$ and i receives a self-charge. Let τ be the moment that ADV starts j. If $\tau < S_i = S_i(C_i)$ then job j, which is pending at time S_i (by Lemma 5), would be chosen instead of job i since $w_j > w_i$. Hence, $\tau \geq S_i$.

1. j **is not urgent at** τ
   As j is still pending at completion time of i, $w_{i'} \geq w_j$. Hence, the charge that i receives in this case is $2w_i - 2w_{f(i)} + w_i + w_j \leq 3w_i - 2w_{f(i)} + w_{i'}$.
2. j **is urgent at** τ
   Since i receives a self-charge, $d_i \geq \tau + p + q_i$, i.e., ALG can complete j first and i later. As j is urgent and $w_j > \beta w_i$, i would have been interrupted by j according to condition [$B2$], contradicts that the phase is of type 1.

□

Lemma 8. *If the phase of job i is of type 2 then the charge that i receives is at most* $\max\left\{\frac{6}{\beta} - 1, 4\right\} \cdot w_i - 2w_{f(i)} + 2w_{i'}$.

Proof. Let i_1 be the job interrupted by i. By observation, before i_1 is either a job completion or a $B1$-interruption. Then, by Lemma 6, the charge from marked jobs of all intervals but $(1,i),(0,i)$ to job i is at most $2w_{i_1} - 2w_{f(i)}$. Let j_1 and j be marked jobs of intervals $(1,i)$ and $(0,i)$, respectively. As i interrupts i_1 by condition [$B2$], at time C_i job i_1 is still pending, so at that moment, ALG will schedule some job i' such that $w_{i'} \geq w_{i_1}$. Note also that $w_{j_1} < 2w_{i_1}$ since otherwise, job j_1 would interrupt job i_1 by condition [$B1$].

As i interrupt i_1 according to [$B2$], $d_i < S_i + p + q_{i_1}(S_i) < S_i + 2p$. If i receives a self-charge, then the starting time of j by ADV is before S_i since otherwise ADV cannot completes job i before $S_i + 2p$ — that violates the i's deadline. Therefore, $w_j \leq w_i$. The charge that i receives is bounded by $2w_{i_1} - 2w_{f(i)} + w_{j_1} + w_j + w_i \leq 4w_i - 2w_{f(i)} + 2w_{i'}$ because $w_{j_1} \leq 2w_{i_1} \leq 2w_i$. If i receives no self-charge, then the total charge of i is at most $2w_{i_1} - 2w_{f(i)} + w_{j_1} + w_j$. In the following, we prove $2w_{i_1} + w_{j_1} + w_j \leq \max\left\{\frac{6}{\beta} - 1, 4\right\} \cdot w_i + 2w_{i'}$.

If $w_j \leq 2w_i$ then the term is straightforwardly bounded by $4w_i + 2w_{i'}$. Consider the case $w_j > 2w_i$. Let τ be the moment that ADV starts j. Then $\tau \in [S_i, C_i)$. If j is not urgent at τ then $w_{i'} \geq w_j$, so $2w_{i_1} + w_{j_1} + w_j < 4w_i + 2w_{i'}$. In the remaining, $w_j > 2w_i$ and j is urgent at time τ.

1. **There exists job** ℓ **such that** $S_i + p + q_\ell \leq d_\ell < \tau + p + q_\ell$ **and** $w_\ell \geq w_i$**.**
   In this case, we have $w_{i'} \geq w_\ell \geq w_i$. If $w_{j_1} \leq w_i$ then $w_{i'} \geq w_{j_1}$. If $w_{j_1} > w_i$ then j_1 is not urgent at its starting time by the ADV since otherwise, j_1 would have interrupted i_1 according to [$B2$]. Hence, j_1 is pending at S_i. By the job selection in condition [$B2$], j_1 is also pending at time C_i. Thus, $w_{i'} \geq w_{j_1}$. Besides, $w_j \leq \max\{2w_i, 4w_{i_1}\}$ since otherwise, j would have interrupted i by [$B1$]. Therefore, $w_j \leq 4w_{i_1}$. We have:
   $$2w_{i_1} + w_{j_1} + w_j \leq 6w_{i_1} + w_{i'} \leq (6/\beta - 1) \cdot w_i + 2w_{i'}$$
2. **There does not exist job** ℓ **such that** $S_i + p + q_\ell \leq d_\ell < \tau + p + q_\ell$ **and** $w_\ell \geq w_i$**.**
   As j is urgent, $w_j < 2w_i + w_{i_1}$ since otherwise j would have interrupted job i by [$B3$]. Hence,
   $$2w_{i_1} + w_{j_1} + w_j < 2w_{i_1} + w_{j_1} + 2w_i + w_{i_1}$$

If $d_{j_1} \geq S_i + p + q_{j_1}$ then $w_{i'} \geq \max\{w_{i_1}, w_{j_1}\}$, so $2w_{i_1} + w_{j_1} + 2w_i + w_{i_1} < 4w_i + 2w_{i'}$ (note that $w_i > w_{i_1}$). If $S_i + q_{j_1} \leq d_{j_1} < S_i + p + q_{j_1}$ and as j_1 is not completed before by ALG then by the choice of scheduled job at interruption [$B2$], $w_{j_1} \leq w_i$. If $d_{j_1} < S_i + q_{j_1}$ then $d_{j_1} < S_{i_1} + k + q_{j_1}$, i.e. j_1 is urgent. In this case, $w_{j_1} < w_i$ since otherwise j_1 would have interrupted i_1 according to [$B2$]. Therefore, we also have $2w_{i_1} + w_{j_1} + 2w_i + w_{i_1} \leq 4w_i + 2w_{i'}$. □

Lemma 9. *If the phase of job i is of type 3 then the charge that i receives is at most* $\max\left\{\frac{3\beta+11}{2\beta+1}, 4 + \frac{3-\beta}{5\beta+2}\right\} w_i - 2w_{f(i)} + 2w_{i'}$.

Proof. By observation, there are at most two consecutive $B3$-interruptions. We consider the cases where the end of the phase contains one or two such interruptions.

1. **There is only one $B3$-interruption in the end of the phase.** Let $i_0 (= i), i_1, i_2$ be jobs started by ALG in the phase such that i_0, i_1 interrupt i_1, i_2 according to [$B3$] and [$B2$], respectively. Let j_0, j_1 and j_2 be marked jobs of intervals $(0, i), (1, i)$ and $(2, i)$, respectively. Let t_0, t_1, t_2 be the starting time of jobs j_0, j_1, j_2 by ADV, respectively. By Lemma 6, the charge from marked jobs of all intervals but $(2, i), (1, i), (0, i)$ to job i is at most $2w_{i_2} - 2w_{f(i_0)}$. As job i_0 is new released and urgent at time S_{i_0}, either i_0 receives no self-charge or if i_0 receives a self-charge then $w_{j_0} < w_{i_0}$. In the latter, the charge of i_0 is at most $2w_{i_2} - 2w_{f(i)} + w_{j_2} + w_{j_1} + w_{j_0} + w_{i_0} < 2w_{i_2} - 2w_{f(i)} + w_{j_2} + w_{j_1} + 2w_{i_0}$. In the former, i_0 receives at most $2w_{i_2} - 2w_{f(i)} + w_{j_2} + w_{j_1} + w_{j_0}$. In the following, we argue that $W := 2w_{i_2} + w_{j_2} + w_{j_1} + \max\{w_{j_0}, 2w_{i_0}\} \leq \max\left\{\frac{3\beta+11}{2\beta+1}, 4\right\} w_{i_0} + 2w_{i'}$. Remark that j_0 does not interrupt i_0 according to [$B1$], so

   $$\max\{w_{j_0}, 2w_{i_0}\} \leq \max\{8w_{i_2}, 4w_{i_1}, 2w_{i_0}\} \leq \max\{2, 8/(2\beta+1)\}w_{i_0} \leq 8/(2\beta+1)w_{i_0}$$

   (because $\beta < 3/2$). Sometimes, we only need a weaker inequality

   $$\max\{w_{j_0}, 2w_{i_0}\} \leq \max\{8w_{i_2}, 4w_{i_1}, 2w_{i_0}\} \leq 2w_{i_0} + 2w_{i_2}.$$

   We also use inequalities $w_{j_2} < 2w_{i_2}$ (because j_2 does not interrupt i_2 according to [$B1$]), and $(2\beta + 1)w_{i_2} \leq w_{i_0}$.

   (a) **Case $w_{j_1} > w_{i_0}$.** By condition [$B3$], we have either $d_{j_1} \geq S_{i_0} + p + q_{j_1}$ or $d_{j_1} < S_{i_1} + p + q_{j_1}$ (otherwise, j_1 plays the role of job ℓ at time $t = S_{i_0}$). In the former, $w_{i'} \geq w_{j_1}$. Hence,

   $$\begin{aligned} 4w_{i_0} + 2w_{i'} &> 2w_{i_0} + 2(2w_{i_1} + w_{i_2}) + w_{j_1} > \\ &> (2w_{i_0} + 2w_{i_2}) + w_{j_1} + 2w_{i_2} + 2w_{i_2} > W. \end{aligned}$$

   In the latter, job j_1 is urgent at $\max\{t_1, S_{i_1}\}$. Moreover, we know that no job ℓ satisfying $S_{i_1} + p + q_\ell \leq d_\ell < S_{i_0} + p + q_\ell$ and $w_\ell \geq w_{i_1}$ by definition of [$B3$]. So there is no job ℓ satisfying $S_{i_1} + p + q_\ell \leq d_\ell < \max\{t_1, S_{i_1}\} + p + q_\ell$ and $w_\ell \geq w_{i_1}$. As $w_{j_1} > w_{i_0}$, j_1 would have interrupted i_1 according to condition [$B3$] (contradiction).

(b) **Case $w_{j_1} \le w_{i_0}$ and $w_{j_2} > w_{i_1}$.** Then j_2 is not urgent at t_2 since otherwise, j_2 would have been scheduled instead of i_2. By condition $[B2]$, $d_{j_2} \ge S_{i_1} + p + q_{j_2}$. Furthermore, by definition of condition $[B3]$, we have $d_{j_2} \ge S_{i_0} + p + q_{j_2}$. Hence, $w_{i'} \ge w_{j_2} > w_{i_1} > w_{i_2}$. Therefore,

$$\begin{aligned} 4w_{i_0} + 2w_{i'} &> w_{j_1} + 2w_{i_0} + (2w_{i_1} + w_{i_2}) + w_{j_2} + w_{i_2} > \\ &> w_{j_1} + (2w_{i_0} + 2w_{i_2}) + w_{j_2} + 2w_{i_2} \ge W \end{aligned}$$

(c) **Case $w_{j_1} \le w_{i_0}$ and $w_{j_2} \le w_{i_1}$.** We have:

$$\begin{aligned} W &= w_{j_1} + w_{j_2} + w_{j_0} + 2w_{i_2} < w_{i_0} + w_{i_1} + \frac{8}{2\beta+1} w_{i_0} + 2w_{i_2} < \\ &< w_{i_0} + \left(w_{i_1} + \frac{1}{2} w_{i_2}\right) + \frac{8}{2\beta+1} w_{i_0} + \frac{3}{2(2\beta+1)} w_{i_0} \le \frac{3\beta+11}{2\beta+1} \cdot w_{i_0}. \end{aligned}$$

2. **There are two $B3$-interruptions in the end of the phase**
Let $i_0 (= i), i_1, i_2, i_3$ be jobs started by ALG in the phase such that i_0, i_1, i_2 interrupt i_1, i_2, i_3 according to $[B3]$, $[B3]$ and $[B2]$, respectively. Let j_0, j_1, j_2 and j_3 be marked jobs of intervals $(0, i), (1, i), (2, i)$ and $(3, i)$, respectively. Let t_0, t_1, t_2, t_3 be the starting time of jobs j_0, j_1, j_2, j_3 by ADV, respectively. By Lemma 6, the charge from marked jobs of all intervals but $(3, i), (2, i), (1, i), (0, i)$ to job i is at most $2w_{i_3} - 2w_{f(i)}$. As job i_0 is new released and urgent at time S_{i_0}, either i_0 receives no self-charge or if i_0 receives a self-charge then $w_{j_0} < w_{i_0}$. In the latter, the charge of i_0 is at most $2w_{i_2} - 2w_{f(i)} + w_{j_3} + w_{j_2} + w_{j_1} + w_{j_0} + w_{i_0} < 2w_{i_2} - 2w_{f(i)} + w_{j_2} + w_{j_1} + 2w_{i_0}$. In the former, i_0 receives at most $2w_{i_2} - 2w_{f(i)} + w_{j_3} + w_{j_2} + w_{j_1} + w_{j_0}$. In the following, we argue that $W' := 2w_{i_3} + w_{j_3} + w_{j_2} + w_{j_1} + \max\{w_{j_0}, 2w_{i_0}\} \le \max\left\{\frac{3\beta+11}{2\beta+1}, 4 + \frac{3-\beta}{5\beta+2}\right\} w_{i_0} + 2w_{i'}$.
The cases will be proceeded similarly as previous. Remark that j_0 does not interrupt i_0 according to $[B1]$, so $\max\{w_{j_0}, 2w_{i_0}\} \le \max\{2w_{i_0}, 4w_{i_1}, 8w_{i_2}, 16w_{i_3}\} \le 2w_{i_0} + 2w_{i_2}$. Similarly, $w_{j_2} \le \max\{2w_{i_2}, 4w_{i_3}\}$ and $w_{j_3} \le 2w_{i_3}$.

(a) **Case $w_{j_1} > w_{i_0}$.** By condition $[B3]$, we have either $d_{j_1} \ge S_{i_0} + p + q_{j_1}$ or $d_{j_1} < S_{i_1} + p + q_{j_1}$. In the former, $w_{i'} \ge w_{j_1}$. Hence,

$$\begin{aligned} 4w_{i_0} + 2w_{i'} &> 2w_{i_0} + w_{j_1} + 3(2w_{i_1} + w_{i_2}) \\ &> (2w_{i_0} + 2w_{i_2}) + w_{j_1} + (2w_{i_2} + 2w_{i_3}) + 2w_{i_3} + 2w_{i_3} > W' \end{aligned}$$

In the latter, job j_1 is urgent at $\max\{t_1, S_{i_1}\}$. Moreover, we know that no job ℓ satisfying $S_{i_1} + p + q_\ell \le d_\ell < S_{i_0} + p + q_\ell$ and $w_\ell \ge w_{i_1}$ by definition of $[B3]$. So there is no job ℓ satisfying $S_{i_1} + p + q_\ell \le d_\ell < \max\{t_1, S_{i_1}\} + p + q_\ell$ and $w_\ell \ge w_{i_1}$. As $w_{j_1} > w_{i_0}$, j_1 would have interrupted i_1 according to condition $[B3]$ (contradiction).

(b) **Case $w_{j_1} \le w_{i_0}$ and $w_{j_2} > w_{i_1}$.** Then j_2 is not urgent at t_2 since otherwise, j_2 would have been scheduled instead of i_2. By condition $[B2]$, $d_{j_2} \ge S_{i_1} + p + q_{j_2}$. Furthermore, by definition of condition $[B3]$, we have $d_{j_2} \geq S_{i_0} + p + q_{j_2}$. Hence, $w_{i'} \geq w_{j_2} > w_{i_1} > w_{i_2}$. Therefore,

$$\begin{aligned} 4w_{i_0} + 2w_{i'} &> w_{j_1} + 2w_{i_0} + (2w_{i_1} + w_{i_2}) + w_{j_2} + w_{i_1} > \\ &> w_{j_1} + 2w_{i_0} + w_{i_2} + w_{j_2} + 3(2w_{i_2} + w_{i_3}) > \\ &> w_{j_1} + (2w_{i_0} + 2w_{i_2}) + w_{j_2} + w_{j_3} + 2w_{i_3} \geq W' \end{aligned}$$

(c) **Case $w_{j_1} \leq w_{i_0}$ and $w_{j_2} \leq w_{i_1}$.** We have:

$$\begin{aligned} W' = w_{j_0} + w_{j_1} + w_{j_2} + w_{j_3} + 2w_{i_3} &\leq (2w_{i_0} + 2w_{i_2}) + w_{i_0} + w_{i_1} + 4w_{i_3} \\ &\leq 3w_{i_0} + 2w_{i_1} + 3w_{i_3} \leq 4w_{i_0} + (3-\beta)w_{i_3} \leq \left(4 + \frac{3-\beta}{5\beta+2}\right) w_{i_0} \end{aligned}$$

where the last inequality is due to $w_{i_3} \leq \frac{1}{5\beta+2} w_{i_0}$ (by combining $w_{i_0} \geq 2w_{i_1} + w_{i_2}$, $w_{i_1} \geq 2w_{i_2} + w_{i_3}$ and $w_{i_2} \geq \beta w_{i_3}$). □